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Regulating Activities with Catastrophic Environmental Effects[1]

M. L. CROPPER

Department of Economics, University of California, Riverside, California 92502

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In this paper a catastrophe is an unforeseen event which reduces society's level of consumption to zero. Two types of catastrophe are analyzed. In one case catastrophe results in a temporary reduction in utility; in the other, catastrophe is irreversible and is tantamount to truncating the planning horizon. The first case characterizes certain types of pollution problems such as radioactive pollution produced by a nuclear power plant. An example of an irreversible catastrophe is the inadvertent depletion of a nonrenewable resource for which no substitute is available.

0. INTRODUCTION

0.1. The central question addressed in this paper is at what rate society should consume a good, such as nuclear energy, which may have catastrophic effects on the environment. This problem is closely related to the broader problem of pollution control, but, for reasons explained below, it cannot be handled within the framework of existing pollution control models.

In most pollution control models, e.g., [2, 6, 9], effluents are discharged as byproducts of consumption or production activity and enter society's utility function as a stock or as a flow. This is a reasonable approximation to reality if the effects of pollution can be regarded as certain and if the disutility caused by pollution varies in a smooth and continuous way with the size of the pollution stock. There are, however, environmental problems in which the effects of pollution are neither certain nor gradual. For example, radioactive waste produced by a nuclear power plant is relatively harmless to humans until it reaches a sufficiently high level. Once this level is reached, however, radionuclides will cause death or severe multilation. The pollution control problem associated with nuclear power plants is complicated by the fact that the threshold concentration of radionuclides must be regarded as uncertain.

Similar circumstances characterize pollution of the stratosphere by aerosols or by supersonic aircraft. When aircraft fly in or through the stratosphere effluents are discharged which erode the ozone layer. The consequences of ozone depletion, like the effects of radioactive pollution, exhibit sharp discontinuities. Substantial ozone depletion is likely to cause severe climatic changes—drought or a significant change in temperature—while small amounts of ozone depletion can be expected to have little

[1] I would like to acknowledge helpful comments from B. A. Forster, R. Schmalensee, and especially from H. Y. Wan, Jr. The responsibility for any errors is, of course, my own.

impact on the environment.[2] Unfortunately, what constitutes a critical level of ozone depletion is highly uncertain; hence, the disastrous consequences of ozone depletion will occur suddenly if they occur at all.

0.2. In the ozone depletion problem and in the case of radioactive pollution there is an uncertain pollution threshold which, if exceeded, will result in drastic losses to society. In this paper we shall refer to the crossing of this pollution threshold as a catastrophe because of the large losses which it entails and because of the suddenness with which it must occur. To capture the notion that catastrophe entails large losses, we suppose that whenever the pollution stock exceeds its critical value, society receives the same level of utility as it would receive if consumption were zero. Hence a catastrophe is defined here as an uncertain event which reduces society's consumption (utility) to zero.

When catastrophe is defined in this way, the range of activities having catastrophic effects is substantially broader than the pollution problems referred to above. For example, inadvertently depleting a nonrenewable resource can be considered a catastrophe since consumption of the resource becomes zero once the amount extracted exceeds the total amount available. If the size of the resource stock is uncertain, then society's optimal depletion problem is analogous to the catastrophe problem discussed above.[3]

In what follows we focus on two problems involving catastrophe. Section I analyzes the steady-state equilibria and approach paths for pollution when the effects of pollution are catastrophic. In Section II, a slightly altered version of the model of Section I is used to derive the optimal rate of extraction of a nonrenewable resource when available reserves are uncertain.

These problems are interesting not only because they are descriptively different from standard resource allocation problems, but because they possess solutions which differ significantly from solutions to analogous problems in which catastrophic effects are absent. In pollution control models [6, 9] where the pollution stock enters the utility function with negative marginal utility, one usually obtains as a solution a unique steady-state equilibrium. When the pollution stock influences utility indirectly by influencing the probability of disaster, a unique saddlepoint is still possible; however, multiple equilibria are more likely to occur and it is even possible that no equilibrium exists.

In optimal depletion problems, it is well known that a planner with a fixed stock of some resource, a positive discount rate, and a strictly concave utility function will decrease his consumption of the resource over time [4, Chap. 13]. However, when the available stock of the resource is uncertain, it may be optimal to increase consumption of the resource over time. It is also possible that society will deplete the resource faster when reserves are uncertain than when they are perfectly known.

[2] Industrial pollutants such as asbestos also exhibit threshold effects, insofar as a certain critical amount of these pollutants must be ingested before death occurs. However, the distribution of costs and benefits in the case of asbestos differs significantly from the examples cited in the text. It is generally true that the individuals who benefit from the use of aerosol products or nuclear energy are the same individuals who suffer the effects of pollution. This is clearly not the case with industrial pollutants, where workers bear a disproportionate share of the costs of pollution.

[3] There is, however, an important difference between catastrophic pollution problems and inadvertent depletion of a nonrenewable resource. In the former case, catastrophe is reversible since it is (usually) possible to reduce the size of the pollution stock. In the latter case catastrophe is irreversible: if the amount extracted exceeds available reserves at t', it will exceed available reserves for all $t > t'$.

I. CATASTROPHIC POLLUTION

I.1. In the model of Section I society consumes a good (nuclear energy) whose production increases the stock of some pollutant (radionuclides). Once created, pollution can be destroyed by devoting resources to pollution control or by allowing the pollution stock to decay naturally. We assume that consumption increases the stock of pollution at an increasing rate, according to the function $g(C)$, while E, expenditure on pollution control, decreases the pollution stock at a decreasing rate, according to the function $h(E)$. The functions $g(C)$ and $h(E)$ have the properties

$$\begin{aligned} &g(0) = 0, \quad g'(C) > 0, \quad g''(C) > 0, \\ &h(0) = 0, \quad h'(E) > 0, \quad h''(E) < 0, \quad \lim_{E \to 0} h'(E) = \infty. \end{aligned} \tag{1.1}$$

[4]

The pollution stock, P, decays naturally at the constant exponential rate α.

Since the amount spent on pollution control and on consumption must equal ϕ, total resources available, the net change in the pollution stock can be written

$$\dot{P} = Z(C) - \alpha P \tag{1.2}$$

where

$$\begin{aligned} Z(C) &= g(C) - h(\phi - C) \\ Z'(C) &= g'(C) + h'(\phi - C) > 0 \\ Z''(C) &= g''(C) - h''(\phi - C) > 0. \end{aligned}$$

[5]

At any time, whether catastrophe occurs or not depends on the size of the pollution stock, P, relative to the critical pollution level, P^*. In many problems involving catastrophe, P^* must be regarded as random. What constitutes a lethal dose of radiation varies from one individual to another. Similarly, the stock of effluents necessary to cause a critical change in the earth's surface temperature will vary depending on conditions in the stratosphere which are outside of the decision maker's control. It is therefore assumed that P^* is a random variable distributed over the interval $[0, \infty)$ with probability density function $f(P^*)$. P^* is assumed to be identically and independently distributed through time. The randomness in P^* should, for the purposes of the model, be interpreted as objective uncertainty rather than as a reflection of ignorance on the part of the decision maker.

In the absence of catastrophe, society's welfare is assumed to be an increasing, concave function of aggregate consumption,

$$U(C) \tag{1.3}$$

with

$$U(0) = 0, \quad U'(C) > 0, \quad U''(C) < 0, \quad \text{and} \quad \lim_{C \to 0} U'(C) = \infty.$$

Equation (1.3) implies that the pollution stock does not enter the utility function directly; instead, P affects utility indirectly by altering the probability that catastrophe

[4] These are essentially the assumptions made by Forster [2].

[5] Unless further restrictions are imposed, $Z(C)$ implies that if C is small enough, a net reduction in pollution is possible. For certain types of pollution problems this assumption is appropriate. However, in some of the pollution problems considered above (for example, the SST) it is the case that once created, pollution cannot be destroyed except by natural processes. In these cases, given the resource constraint, there is some level of consumption C_0 below which no net pollution is created. Above C_0 net pollution creation is positive. If this is the case, the function $Z(C)$ can still be used to describe pollution creation, provided we constrain $C \geq C_0$.

occurs. This assumption is made so that we may focus on the catastrophic effects of pollution. The consequences of allowing P to enter the utility function directly are explored in I.3 below.

If at any t, $P < P^*$, then society's welfare is described by (1.3). On the other hand, if catastrophe occurs, $U(t) = 0$. This implies that in terms of utility, disaster is equivalent to a zero level of consumption. Intuitively, this concurs with the notion that catastrophe is the worst that could possibly happen.

Since catastrophe is a random event, expected utility at any instant is given by

$$\int_P^\infty U(C)f(P^*)dP^* = \Lambda(P)U(C), \tag{1.4}$$

where

$$\Lambda(P) \equiv \int_P^\infty f(P^*)dP^*$$

is the probability that catastrophe will *not* occur, given that the size of the pollution stock is P.[6] Since $U(t) = 0$ whenever $P \geq P^*$, Eq. (1.4) holds for all t and society will seek to maximize

$$J[C] = \int_0^\infty e^{-\delta t}\Lambda(P)U(C)dt \tag{1.5}$$

subject to

$$\dot{P} = Z(C) - \alpha P, \qquad C \geq 0,$$
$$P \geq 0, \qquad \text{and} \qquad P(0) = P_0$$

where δ is the social rate of discount.

Equation (1.3) guarantees that the constraint $C \geq 0$ is always satisfied with strict inequality. If we assume, as seems reasonable, that $\lim_{P^* \to 0} f(P^*) = 0$, then it can be shown (see below) that P is always positive along an optimal trajectory. In view of these considerations, the Hamiltonian for the problem will be written

$$H = \Lambda(P)U(C) + \Psi[Z(C) - \alpha P], \tag{1.6}$$

where Ψ is the co-state variable associated with the pollution stock. The necessary conditions which must be satisfied along an optimal path are

$$\Lambda(P)U'(C) + \Psi Z'(C) = 0, \tag{1.7}$$

$$\dot{\Psi} = (\alpha + \delta)\Psi - \Lambda'(P)U(C), \tag{1.8}$$

$$\dot{P} = Z(C) - \alpha P, \tag{1.9}$$

$$\lim_{t \to \infty} e^{-\delta t}\Psi(t)P(t) = 0.\text{[7]} \tag{1.10}$$

The co-state variable Ψ may be interpreted as the decrease in future utility caused by an increase in the pollution stock. Since $Z'(C)$ is the change in the pollution stock resulting from an additional unit of consumption, Eq. (1.7) states that the expected gain in utility from consuming an additional unit of nuclear energy must just equal

[6] Equation (1.4) is consistent with a state-of-nature framework in which the decision maker can influence the probability that a particular state (catastrophe/no catastrophe) will occur.

[7] Scheinkman and Benveniste [10] have shown that (1.10) is a necessary condition, provided $\delta > 0$, even if the planning horizon is infinite.

the decrease in future utility which that consumption entails. Equations (1.8) and (1.9) are equations of motion, while (1.10) is the transversality condition.[8]

I.2. The model of (1.5), although fairly simple, is quite rich in possible solutions. Figures 1 through 3 illustrate alternate equilibria and approach paths for the system under various assumptions about the shapes of the stationary loci $\dot{P} = 0$ and $\dot{\Psi} = 0$. Previous assumptions guarantee that the stationary locus $\dot{P} = 0$ is always positively sloped,

$$\left.\frac{d\Psi}{dP}\right|_{\dot{P}=0} = \frac{-(Z'C_P - \alpha)}{Z'C_\Psi} > 0, \tag{1.11}$$

where

$$C_P = \frac{-\Lambda' U'}{\Lambda U'' + \Psi Z''} < 0$$

and

$$C_\Psi = \frac{-Z'}{\Lambda U'' + \Psi Z''} > 0.$$

Furthermore, $\dot{P} = 0$ must intersect the Ψ-axis at the point $-U(C_0)/Z(C_0)$ since

$$\lim_{P\to 0} \Psi|_{\dot{P}=0} = \frac{\Lambda(0)U'(C_0)}{Z'(C_0)} = \frac{-U'(C_0)}{Z'(C_0)}, \tag{1.12}$$

C_0 being the level of consumption for which $Z(C) = 0$. The limit of Ψ as $P \to \infty$ is given by

$$\lim_{P\to\infty} \Psi|_{\dot{P}=0} = \lim_{P\to\infty} \frac{-\Lambda(P)U'(C)}{Z'(C)} = 0. \tag{1.13}$$

Along $\dot{\Psi} = 0$, $\Psi = \Lambda'(P)U(C)/(\alpha + \delta)$, hence the locus $\dot{\Psi} = 0$ will intersect the P-axis at the end-points of the $f(P^*)$ distribution.[9] Unfortunately, the slope of $\dot{\Psi} = 0$

$$\left.\frac{d\Psi}{dP}\right|_{\dot{\Psi}=0} = \frac{\Lambda'' U + \Lambda' U' C_P}{\alpha + \delta - \Lambda' U' C_\Psi} \tag{1.14}$$

is ambiguous since $\Lambda''(P) = -f'(P)$ may be either positive or negative.

To simplify matters somewhat, we shall suppose that the density function $f(P^*)$ is unimodal with $f'(P^*) \geq 0$ for $0 \leq P^* \leq \bar{P}$ and $f'(P^*) < 0$ whenever $P^* > \bar{P}$, $\bar{P}$ being the mode of the distribution. This assumption guarantees that $d\Psi/dP|_{\dot{\Psi}=0}$ is unambiguously positive to the right of $\bar{P}$. To the left of $\bar{P}$ the slope of $\dot{\Psi} = 0$ is of the same sign as $\Lambda''U + \Lambda'U'C_P$. Figures 1–3 are drawn on the assumption that $\Lambda''U + \Lambda'U'C_P < 0$ to the left of $\bar{P}$.[10] This implies that $\dot{\Psi} = 0$ is U-shaped with the shadow price of pollution increasing above $\dot{\Psi} = 0$ and decreasing below $\dot{\Psi} = 0$.

The optimal equilibria and approach paths for the system (1.7)–(1.10) depend critically on the behavior of Ψ, the shadow price of pollution, as $P \to \infty$. Figure 1 is drawn on the assumption that as $P \to \infty$, $\Psi \to 0$ faster along $\dot{P} = 0$ than along $\dot{\Psi} = 0$.

[8] Equations (1.7)–(1.10) provide necessary but not sufficient conditions for the optimal path of extraction. Given the nature of the instantaneous utility function $\Lambda(R)U(C)$, joint concavity is clearly not an appropriate assumption. Hence one cannot guarantee that the usual sufficiency conditions are satisfied.

[9] Figures 1–3 are drawn on the assumption that $f(P^*) = 0$ for $P^* > \hat{P}$. If it were the case that $f(P^*)$ were positive for $0 < P^* < \infty$ then $\dot{\Psi} = 0$ would approach 0 asymptotically as $P \to \infty$.

[10] This guarantees that the sufficiency conditions hold to the left of $\bar{P}$.

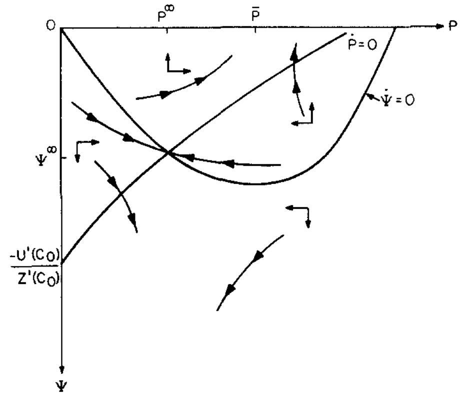

FIG. 1. The single equilibrium case.

In this case, the system may have a unique stable equilibrium, as pictured in Fig. 1. However, this is not the only possible solution. When $\dot{\Psi} = 0$ intersects the P-axis to the left of $\dot{P} = 0$, the situation pictured in Fig. 2 may occur. Figure 2 poses an interesting problem for planners. There are in this case two possible equilibria: a stable equilibrium at $(P_1^\infty, \Psi_1^\infty)$ with relatively low levels of pollution and consumption, and an unstable focus point $(P_2^\infty, \Psi_2^\infty)$ with consumption and pollution both higher than at $(P_1^\infty, \Psi_1^\infty)$. Planners must choose between these two points, possibly by comparing welfare integrals along paths leading to $(P_1^\infty, \Psi_1^\infty)$ and $(P_2^\infty, \Psi_2^\infty)$, and their choice will depend critically on P_0.

This situation may be contrasted with the relatively simple solutions obtained by Forster [2] and Plourde [9] in models where utility is a function of both consumption and pollution and no catastrophe is possible. In the Forster and Plourde models there is always a unique, stable equilibrium point (P^∞, Ψ^∞). In the present model, multiple equilibria may occur because, for $P > \bar{P}$, increases in P increase the probability of disaster at a *decreasing* rate. Hence, for some values of (P, Ψ) it will not pay to reduce the pollution stock further, and movement to $(P_1^\infty, \Psi_1^\infty)$ is not, therefore, optimal.

It is also possible, as illustrated in Fig. 3, that there is no steady-state equilibrium solution satisfying Eqs. (1.7)–(1.10). In the situation pictured in Fig. 3, paths originat-

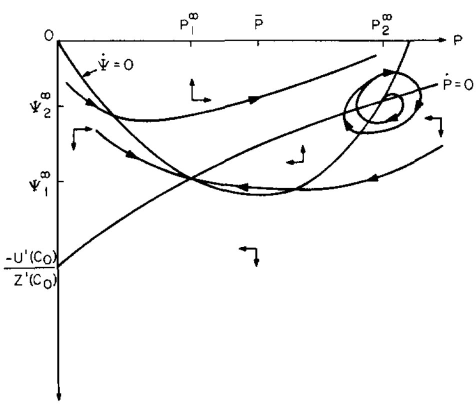

FIG. 2. The case of multiple equilibria.

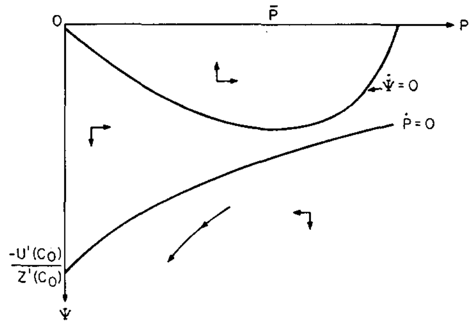

FIG. 3. The no-equilibrium case.

ing below the line $\dot{P} = 0$ will move in the direction of the Ψ-axis; however, it can be shown that $P^\infty = 0$ violates the necessary conditions for optimality, and hence, that no equilibrium exists.

If $P = 0$, then for the constraint $P \geq 0$ to be satisfied it must be the case that $\dot{P} \geq 0$. With this constraint, the Hamiltonian becomes

$$H = \Lambda(P)U(C) + (\Psi + \pi)[Z(C) - \alpha P], \tag{1.15}$$

where, π, $\pi(t) \geq 0$, is the multiplier associated with $Z(C) - \alpha P \geq 0$. The necessary conditions, evaluated at $P = 0$, are given by

$$U'(C_0) + (\Psi + \pi)Z'(C_0) = 0 \tag{1.16}$$

$$\dot{\Psi} = (\alpha + \delta)\Psi - \Lambda'(0)U(C_0) + \alpha\pi. \tag{1.17}$$

If the probability density of P^* is zero when $P^* = 0$, (1.17) reduces to

$$\dot{\Psi} = (\alpha + \delta)\Psi + \alpha\pi. \tag{1.18}$$

Suppose that $P^\infty = 0$. Then to find the corresponding equilibrium value of the co-state Ψ, (1.16) and (1.18) must be solved simultaneously. Substituting from (1.18) into (1.16) implies that $\Psi = \alpha U'(C_0)/[\delta Z'(C_0)]$; however, this contradicts Eq. (1.16) which implies that $\Psi \leq -U'(C_0)/Z'(C_0) < 0$ along an optimal trajectory. Hence, in the situation illustrated in Fig. 3, no steady-state equilibrium exists.

Figures 1–3 are drawn on the assumption that $\Lambda''U + \Lambda'U'C_P < 0$. If it is the case that $\Lambda''U + \Lambda'U'C_P$ is alternately positive and negative when $P < \bar{P}$, the situation is considerably more complicated and more than two equilibria are possible.

I.3. The preceding analysis is based on the assumption that below the critical level P^* the pollution stock does not influence utility at all. While this may be an acceptable simplifying assumption, it is nevertheless of interest to know how steady-state equilibria and approach paths are altered if the pollution stock enters the utility function directly.

Suppose that the utility function $U(C)$ is replaced by

$$U(C, P) \tag{1.19}$$

where

$$U_C > 0, \qquad U_{CC} < 0, \qquad U_{CP} \leq 0, \qquad U_P < 0, \qquad U_{PP} < 0,$$

and

$$\lim_{C \to 0} U_C(C, P) = \infty \quad \text{for all } P$$

$$\lim_{P \to 0} U_P(C, P) = 0 \quad \text{for all } C,$$

If, as above, catastrophe is to be equivalent in utility terms to a zero level of consumption, then utility is equal to $U(0, P)$ whenever $P \geq P^*$. If catastrophe does not occur, then utility is given by (1.19). In view of these assumptions expected utility becomes

$$\int_0^P U(0, P)f(P^*)dP^* + \int_P^\infty U(C, P)f(P^*)dP^*, \tag{1.20}$$

and the planning authority will seek to maximize

$$\int_0^\infty e^{-\delta t} \left\{ \int_0^P U(0, P)f(P^*)dP^* + \int_P^\infty U(C, P)f(P^*)dP^* \right\} dt \tag{1.21}$$

subject to

$$\dot{P} = Z(C) - \alpha P \quad \text{and} \quad P(0) = P_0.$$

The expression for expected utility (1.20) can be simplified as in Eq. (1.4) above; however, the Hamiltonian function and necessary conditions corresponding to (1.21) are considerably more complicated than (1.7)–(1.10) above. We shall, for this reason, state the properties of the stationary loci and approach paths of the model without supplying mathematical details.

In the present case it can be shown that the locus $\dot{P} = 0$ is everywhere positively sloped and has the properties pictured in Figs. 1–3. Determining the shape of $\dot{\Psi} = 0$ is, however, difficult and all that can be said, in general, is that

$$\lim_{P\to 0} \Psi|_{\dot{\Psi}=0} = 0 \quad \text{and} \quad \lim_{P\to\infty} \Psi|_{\dot{\Psi}=0} = -\infty. \tag{1.22}$$

With only this restriction, it is possible to obtain a unique equilibrium similar to that shown in Fig. 1; however, multiple equilibria, illustrated in Fig. 4, are likely to occur. Since multiple equilibria similar to those in Fig. 4 are possible in the simpler case discussed above, allowing pollution to influence utility directly does not appear to alter qualitatively the solution to the pollution control problem. The only case which is no longer possible is the no-equilibrium solution pictured in Fig. 3. This is ruled out by virtue of (1.22) and the shape of $\dot{P} = 0$. Hence inserting P in the utility function guarantees that there exists at least one equilibrium solution to the pollution control problem.

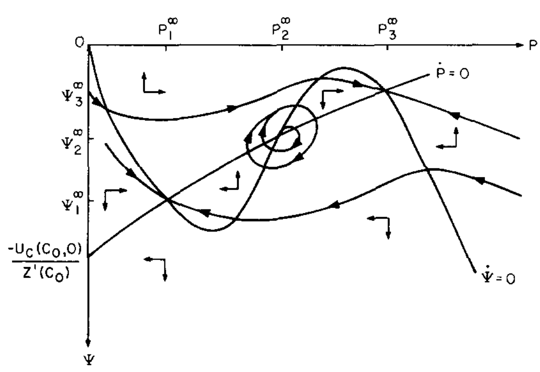

FIG. 4. Possible equilibria when $U = U(C, P)$.

II. DEPLETION WITH UNCERTAIN RESERVES

II.1. We now consider a situation in which the occurrence of catastrophe reduces the level of consumption permanently rather than temporarily. Suppose that society possesses a fixed supply of a nonrenewable resource, and suppose that the resource is consumed at the rate $C(t)$. As soon as the total amount of the resource extracted exceeds the size of the resource stock, no more of the resource can be consumed and utility will remain at zero forever after.[11] If the size of the resource stock is uncertain, then society's optimal depletion problem is analogous to the catastrophe problem studied in Section I. With appropriate modifications, therefore, the model of Section I can be used to determine the optimal pattern of resource depletion when the stock of a nonrenewable resource is uncertain.[12]

Suppose that society possesses an amount S of a nonrenewable resource, where S is a random variable distributed over the interval $[0, \infty)$ with probability density function $f(S)$. For convenience, we shall suppose that $f(S)$ is strictly positive over the interval $[0, \infty)$. Let S be depleted at the rate $C(t)$ and define $R(t) = \int_0^t C(\tau)d\tau$ to be the amount of the resource mined as of time t. If at t, $R(t) < S$, then exhaustion has not yet occurred and utility is given by $U(C)$, where $U(C)$ has the properties (1.3) and

$$\lim_{C \to 0} \eta(C) > 0, \tag{2.1}$$

$\eta(C) \equiv -CU''/U'$. If, on the other hand, $R(t) \geq S$, $C = 0$ and $U(C) = 0$, as well.

In view of these assumptions, expected utility at any instant may be written

$$\int_R^\infty U(C)f(S)dS = \Lambda(R)U(C), \tag{2.2}$$

where $\Lambda(R) \equiv \int_R^\infty f(S)dS$ is the probability that exhaustion has not yet occurred, given that R of the resource has been mined to date. Expected utility is therefore equivalent to certain utility multiplied by $\Lambda(R)$, which may be interpreted as a discount rate which is increasing over time.[13] Equation (2.2) is also consistent with a state-of-nature framework in which the decision maker can influence the probability that a particular state (exhaustion/no exhaustion) will occur.

Society is assumed to maximize discounted expected utility of consumption,

$$\int_0^\infty e^{-\delta t}\Lambda(R)U(C)dt, \tag{2.3}$$

[11] This assumes, of course, that no substitute for the resource is available. However, even if substitutes may become available, it is possible for society to deplete the nonrenewable resource before the substitutes are discovered.

[12] This problem has been studied in a more general context by Kemp [7]. Kemp obtains an equation similar to our (2.12); however, he does not derive properties of the optimal path of extraction for particular classes of utility functions.

[13] This is similar to the results obtained in other dynamic allocation problems in which the length of the planning horizon is uncertain (see e.g., Yaari [12], Heal [4], and Heal and Dasgupta [1]). In such cases one generally obtains the result that maximizing expected utility over an uncertain horizon is equivalent to maximizing utility over a certain horizon with the inclusion of an additional discount term in the maximand.

subject to

$$\dot{R} = C \qquad R_0 = 0 \qquad C(t) \geq 0.$$[14]

The Hamiltonian function corresponding to this problem may be written (omitting the discount factor) as

$$H = \Lambda(R)U(C) + pC + rC, \tag{2.4}$$

where $p(t)$ is the co-state variable associated with $\dot{R} = C$ and $r(t)$ is a multiplier associated with $C(t) \geq 0$. Along an optimal path $C(t)$ must maximize (2.4) for given p, t, and R, implying

$$\Lambda(R)U'(C) + p + r = 0, \tag{2.5}$$

and, in addition,

$$\dot{R} = C \tag{2.6}$$

$$\dot{p} = \delta p - \Lambda'(R)U(C) \tag{2.7}$$

$$r \geq 0, \qquad rC = 0. \tag{2.8}$$

When $\delta > 0$, it has been shown by Scheinkman and Benveniste [10] that

$$\lim_{t\to\infty} e^{-\delta t}p(t)R(t) = 0 \tag{2.9}$$

is also a necessary condition.

Before examining the implications of (2.5)–(2.9) we note that the solution to (2.3), if it exists, yields the optimal path of extraction contingent on the event that the resource has not yet been exhausted. In implementing the plan it is of course possible that exhaustion will unexpectedly occur at some time T and that $C(t)$ will drop to zero thereafter. It should also be noted (see Appendix) that the solution to (2.3) allows for revision of expectations about the size of available reserves based on the observation that exhaustion has not yet occurred.

II.2. Equations (2.5)–(2.9) can be used to derive properties of the optimal path of extraction under uncertainty. What we are interested in is how this path compares with the pattern of extraction in the certainty case. When S is certain, the problem faced by planners is to

$$\max_{C(t)} \int_0^\infty e^{-\delta t}U(C)dt \tag{2.10}$$

subject to

$$\dot{R} = C \qquad R \leq S_0 \qquad C(t) \geq 0.$$

For utility functions satisfying (1.3) and (2.1) and for $\delta > 0$ it is well known (see, for example, [4, Chap. 13]) that it is optimal to exploit the resource over an infinite horizon with the amount extracted declining continuously to zero, according to

$$\dot{C}/C = -\delta/\eta(C). \tag{2.11}$$

[14] Expected utility maximization is not the only criterion which can be used when S is random. It is, for example, possible to maximize discounted utility subject to a probabilistic constraint of the form $P(R \leq S) \geq \lambda$. However, this problem can always be reduced to the problem "maximize discounted utility subject to $R \leq S^*$" where S^* is determined by the condition $P(S \leq S^*) \geq \lambda$. Thus the chance-constrained problem is equivalent to the standard depletion problem with $S = S^*$.

When S is uncertain and $C > 0$, the condition corresponding to (2.11) is

$$\frac{\dot{C}}{C} = \frac{-\delta + \gamma[U/U' - C]^{15}}{\eta(C)} \tag{2.12}$$

where $\gamma \equiv -\Lambda'(R)/\Lambda(R)$ is the conditional probability of exhaustion, given that exhaustion has not yet occurred. The shape of $C(t)$ is now the result of two factors, δ, reflecting a preference for current consumption, and the term $\gamma[U/U' - C]$, which may be interpreted as reflecting a conservation motive.[16] For $U(C)$ of the form (1.3) the latter term is always positive, indicating that exploitation should be postponed to reduce the probability of exhaustion. Equation (2.12) thus implies that due to the possibility of inadvertent depletion the path of extraction decreases more slowly under uncertainty than under certainty. Indeed, depending on the relative strengths of δ and $\gamma[U/U' - C]$, the path of extraction may even increase over time.

If one is willing to place reasonable restrictions on γ, the conditional probability of exhaustion, stronger statements than this can be made. Suppose that γ is a nondecreasing function of R, the amount mined to date. Then since $[U/U' - C]$ is an increasing function of C, it follows that if the path of extraction is increasing at some point in time, it is increasing thereafter. This rules out paths of an inverted U-shape and implies that either:

(i) $C(t)$ is monotone decreasing for all t; or

(ii) $C(t)$ is nondecreasing for some $t = t'$ and for all $t > t'$.

In the special case in which $\gamma(R)$ is constant for all R it follows that the path of extraction is either monotone decreasing or monotone nondecreasing.

An interesting case arises when δ, the social rate of discount, is zero. In the certainty case it is well known that no solution exists for the class of utility functions considered here unless $\delta > 0$ [3, p. 4]. Under uncertainty this is no longer necessarily the case, although one cannot guarantee that a solution exists when $\delta = 0$. (As a counterexample, consider the case of $U(C) = C^{\beta}$, $0 < \beta < 1$, $f(S) = e^{-S}$.)

When $\delta = 0$, (2.12) implies that the optimal path of extraction is monotone increasing over time. Since only a finite amount of the resource exists, there must come a time T when the resource is exhausted and consumption falls to zero. Thus a zero discount rate, which intuitively should result in a more "future-oriented" policy than a positive discount rate, turns out to be a mixed blessing for future generations. Generations in between $t = 0$ and $t = T$ will receive more of the resource when $\delta = 0$; however, generations after time T consume none of the resource and are therefore worse off than if δ were positive.

The preceding results contrast sharply with the optimal rate of extraction when S is certain. There is, however, at least one result from the certainty case which continues to hold under uncertainty. When S is certain and $U(C)$ is of the form (1.3) it can be shown [4, Chap. 13] that society will plan to mine a positive amount of the resource over an infinite horizon. Under the above assumptions this result also holds

[15] Equation (2.12) is obtained by differentiating (2.5) with respect to t and then substituting into the resulting equation from (2.6) and (2.7).

[16] The function $\Lambda(R)U(C) = g(R, C)$ is analogous to Vousden's [11] utility function incorporating a conservation motive. In Vousden's model the conservation motive causes society to deplete the resource at a slower rate, with the path of extraction decreasing more slowly than if no conservation motive were present.

under uncertainty. In order for equation (2.9) to be satisfied it is necessary that $p(t)$ or $e^{-\delta t}$ approach zero as t approaches the date at which planned extraction ceases. Clearly $e^{-\delta t} \to 0$ if $t \to \infty$; however, if planned extraction is increasing (implying $\dot{p} > 0$) then it is possible for $p(t)$ to approach zero in finite time. If $p(t)$ were to approach zero in finite time, however, Eq. (2.5) would be violated. Hence, Eq. (2.9) can be satisfied only if society plans to extract the resource over an infinite horizon.

II.3. The foregoing results can be augmented and illustrated by examining the path of extraction for the class of utility functions having constant elasticity of marginal utility, $U(C) = C^{\beta}$, $0 < \beta < 1$. When S is certain and $U(C) = C^{\beta}$, an optimal path of extraction exists, provided $\delta > 0$, and is given by

$$C(t) = \frac{\delta}{1 - \beta} S_0 e^{-(\delta/1-\beta)t}. \tag{2.13}$$

Equation (2.13) implies that the resource stock is spread over an infinite horizon, with the amount extracted declining continuously to zero at the constant exponential rate $-\delta/1 - \beta$.

When S is uncertain, (2.12) becomes

$$\frac{\dot{C}}{C} = \frac{-\delta}{1 - \beta} + \frac{\gamma C}{\beta}, \tag{2.14}$$

which may be solved to yield the path of extraction as a function of R and the initial amount extracted

$$C(t) = \frac{1}{[C_0^{-1} - b\gamma]e^{t/\beta b} + b\gamma}, \quad b \equiv \frac{1 - \beta}{\delta\beta}. \tag{2.15}$$

If the path (2.15) is to be optimal, C_0 must be appropriately constrained to assure that $C(t) \geq 0$ for all t. We shall assume that γ, the conditional probability of exhaustion, has a maximum, and denote that maximum by $\hat{\gamma}$. The constraint

$$0 < C_0 < \frac{1}{b\hat{\gamma}} \tag{2.16}$$

then guarantees that an optimal path exists.

Several statements may be made about the nature of the optimal path of extraction without specifying exactly the form of $f(S)$. In view of (2.16), C_0 may be written

$$C_0 = \frac{\alpha}{b\hat{\gamma}}, \quad 0 < \alpha \leq 1, \tag{2.17}$$

which implies

$$C(t) = \frac{1}{\gamma b\left(\left[\frac{\hat{\gamma}}{\alpha\gamma} - 1\right] e^{t/\beta b} + 1\right)}. \tag{2.18}$$

Equation (2.18) may be substituted into (2.14) to obtain

$$\frac{\dot{C}}{C} = \frac{-\delta}{1 - \beta}\left[1 - \frac{1}{\left(\frac{\hat{\gamma}}{\alpha\gamma} - 1\right) e^{t/\beta b} + 1}\right]. \tag{2.19}$$

Since $\hat{\gamma}/\alpha\gamma \geq 1$ and $1/\beta b$ is positive, (2.19) implies that C is nonincreasing along an optimal path. The path of extraction will in fact decrease over time, provided that γ, the conditional probability of exhaustion, is an increasing function of R. When this condition obtains the path of extraction declines continuously to zero, as in the certainty case, and total planned extraction is bounded.[17] If, however, the conditional probability of exhaustion remains constant as R increases then planned extraction of the resource remains constant for all t, i.e., $\dot{C} = 0$, and $R(t)$ is unbounded as t approaches infinity. This can occur, for example, if the resource stock has an exponential distribution, $f(S) = e^{-S}$, implying that

$$C(t) = \frac{\delta\beta}{1 - \beta} \tag{2.20}$$

along an optimal path.

It is interesting to note that in the latter case society will always use up any finite amount of the resource faster under uncertainty than under certainty. Thus, even though uncertainty has the effect of lowering the discount rate [see Eq. (2.12)], future generations may be worse off if reserves are viewed as uncertain than if they are treated as certain.

III. CONCLUDING REMARKS

III.1. Many individuals (see, for example, [5]) have stressed the fact that nuclear power plants and stratospheric flight may result in large losses to society, although the probability of extremely large losses is small. The model presented in Section I captures the "small probability of large loss" feature of these pollution problems by assuming that the uncertain effects of pollution are potentially disastrous.

In the model of Section I disaster occurs if the pollution stock, P, exceeds some critical value P^*. The consequence of disaster is to reduce the level of utility to zero so that, in utility terms, catastrophe is equivalent to zero consumption. When the effects of pollution are potentially catastrophic the unique, stable equilibrium which characterizes many pollution control models, e.g., [2, 6, 9], no longer obtains and multiple equilibria, as well as a no-equilibrium solution, are possible. Allowing the pollution stock to enter the utility function directly guarantees the existence of an equilibrium solution but does not rule out the possibility of multiple equilibria.

III.2. The notion of catastrophe developed in the context of pollution control problems is equally applicable to problems of resource depletion when available reserves of an exhaustible resource are considered uncertain. In this situation catastrophe occurs when the total amount of the resource extracted exceeds the uncertain resource stock. When the resource stock is uncertain, optimal depletion of the resource no longer follows the pattern indicated in the certainty case.

In the certainty case society will, for the class of utility functions considered in this paper, deplete the resource over an infinite horizon with the amount extracted declining continuously to zero as $t \to \infty$. When reserves are uncertain the path of planned extraction is no longer necessarily monotonic and may even increase over time. Specifically, if the probability of running out of the resource, given that it has not yet been exhausted, is a nondecreasing function of the amount extracted to date,

[17] The boundedness of $R(t)$ as $t \to \infty$ can be demonstrated by substituting (2.18) into the definition of $R(t)$ and evaluating the resulting integral as a function of γ. Since $\gamma \leq \hat{\gamma}$ for all t, it is possible to establish an upper bound for $R(t)$ and to show that $\lim_{t\to\infty} R(t) < \infty$.

then the path of planned extraction will be either

(i) monotone decreasing for all t; or

(ii) nondecreasing for some $t = t'$ and all for all t thereafter.

For utility functions exhibiting constant elasticity of marginal utility the path of extraction will be of the form (i) if the conditional probability of exhaustion increases as more of the resource is mined. Planned extraction will be constant over time only if the conditional probability of exhaustion remains constant as more of the resource is depleted.

In closing, it is interesting to note that future generations may be worse off under uncertainty than they would be if the size of the resource stock were certain. This is because society will always use up the resource stock faster under uncertainty if the path of extraction is of the form (ii).

APPENDIX

The purpose of this appendix is to show that the optimal path of extraction in Section II allows for revision of expectations about S, the size of the resource stock.[18] The method of proof follows Long [8].

Let $f[S|R(\tau)]$ be the density function over the resource stock, given that $R(\tau)$ of the resource has already been mined and that exhaustion has not yet been observed to occur.

One may likewise define

$$\Lambda[R(t)|R(\tau)] \equiv \int_{R(t)}^{\infty} f[S|R(\tau)]dS \tag{A.1}$$

to be the probability that the resource stock is at least of size $R(t)$, given that $R(\tau)$ has been mined to date.

The question is whether the path $C(t)$ which maximizes

$$\int_{0}^{\infty} e^{-\delta t}\Lambda[R(t)|0]U(C)dt \tag{A.2}$$

subject to

$$\dot{R} = C \qquad R(0) = 0 \qquad C(t) \geq 0 \tag{A.3}$$

is consistent with the path chosen at time t', provided exhaustion has not yet been observed to occur, to maximize

$$\int_{t'}^{\infty} e^{-\delta t}\Lambda[R(t)|R(t')]U(C)dt \tag{A.4}$$

subject to (A.3).

Since $\Lambda[R(t), R(t')|0]$, the joint probability that $S > R(t)$ and $S > R(t')$, equals $\Lambda[R(t)|0]$ for $R(t) > R(t')$, expectations will be revised according to

$$\Lambda[R(t)|R(t')] = \frac{\Lambda[R(t)|0]}{\Lambda[R(t')|0]}. \tag{A.5}$$

[18] In the model of Section I the planner has no need to revise his beliefs about P^*. This is because P^*, unlike S, is truly random. $f(P^*)$ is assumed to be the true, objective distribution over P^*.

It thus follows that, given $R(t')$, the revised probability of exhaustion differs from the initial probability only by a multiplicative constant, and (A.2) and (A.4) will, therefore, yield identical solutions.

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