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On the finite sample properties of pre-test estimators of spatial models[☆]

Gianfranco Piras [a,1], Ingmar R. Prucha [b,*]

[a] Regional Research Institute, West Virginia University, Morgantown, WV 26506, United States
[b] Department of Economics, University of Maryland, College Park, MD 20742, United States

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ABSTRACT

This paper explores the properties of pre-test strategies in estimating a linear Cliff–Ord-type spatial model when the researcher is unsure about the nature of the spatial dependence. More specifically, the paper explores the finite sample properties of the pre-test estimators introduced in Florax et al. (2003), which are based on Lagrange Multiplier (LM) tests, within the context of a Monte Carlo study. The performance of those estimators is compared with that of the maximum likelihood (ML) estimator of the encompassing model. We find that, even in a very simple setting, the bias of the estimates generated by pre-testing strategies can be very large and the empirical size of tests can differ substantially from the nominal size. This is in contrast to the ML estimator. However, if the true data generating process corresponds to the spatial error or lag model the issues arising with the pre-test estimators seem to be lessened.

1. Introduction

The recent literature on spatial econometrics has been concerned with model specification issues both in cross sectional as well as panel data analysis.[2] In the present paper we will focus exclusively on cross sectional models. Empirical work is often based on estimation strategies which entails estimating initially a linear model (i.e., without spatial dependencies), followed by testing for spatial dependences, and re-estimation if spatial dependence cannot be rejected. It is well known that, in general, pre-test strategies may potentially introduce bias for both the parameter estimates and corresponding standard errors.[3] Of course, on the other hand, efficiency may be lost when the researcher estimates a more general model than necessary. The purpose of this paper is to explore the implications of some common pre-test strategies used in the estimation of Cliff–Ord-type spatial models.

In estimating Cliff–Ord-type models two forms of spatial dependences are usually considered in applied work, which correspond to two different model specifications. The first form of spatial dependence relates to the error term and specifies a spatial auto regressive process for the disturbances. Correspondingly, the model that derives from it is often referred to as a spatial error model (see, e.g., Anselin, 1988b). The second form arises when the value of the dependent variable corresponding to each cross-sectional unit is jointly determined with the values at all other neighboring cross-sectional units. This is achieved through the inclusion of a weighted average of the dependent variable which is often described in the literature as a spatial lag. Consequently, the model that derives from it is referred to as a spatial autoregressive model or, simply, a spatial lag model (see, e.g., Anselin, 1988b).

Burridge (1980) and Anselin et al. (1996) propose simple LM diagnostic tests, based on Ordinary Least Squares (OLS) residuals, for spatial error autocorrelation or spatial lag dependence. More recently Florax et al. (2003) suggest a simple selection criterion conditional upon the results of these specification tests. It should be noted that this testing strategy only leads to the estimation of either the spatial lag or the spatial error models and never of a model that contains both error and lag dependences, i.e., of the encompassing or full model. A partial explanation for this may be that the parameters of the full model are not identified if the model does not contain exogenous variables. However, most empirical specifications include exogenous variables, and in this situation the parameters of the full model are identified under mild regularity conditions; see, e.g., Kelejian and Prucha (1998). Thus in this situation the researcher can estimate the parameters of the full model, and is not forced to select either the spatial lag or error model. Since the data may have been generated by the full model, which includes the spatial lag and error model as special cases, this reduces the likelihood of model misspecification. Of course, if the data have been generated by either the spatial lag or error model, estimating the full model will lead to a loss of efficiency. On the other hand, using a pre-testing strategy may yield biased estimates and may result in a situation where the employed asymptotic distribution of

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* Corresponding author. Tel.: +1 301 405 3499.
E-mail addresses: Gianfranco.Piras@mail.wvu.edu (G. Piras), prucha@econ.umd.edu (I.R. Prucha).
[1] Tel.: +1 304 293 2643.
[2] See e.g., Cliff and Ord (1972, 1973, 1981); Florax and Folmer (1992); Anselin et al. (1996); Anselin (1988a); Florax et al. (2003); Baltagi et al. (2007, 2003, 2009, 2008); Baltagi and Liu (2008); Burridge (1980); and Debarsy and Ertur (2010).
[3] See, e.g., Leeb and Poetscher (2008) for recent fundamental results and Judge et al. (1985, Ch. 3) for a classical text book presentation.

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On the finite sample properties of pre-test estimators of spatial models

Gianfranco Piras[1] Ingmar R. Prucha[2]

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[1]Regional Research Institute, West Virginia University, Morgantown, WV 26506, Tel.: +1-304-293-2643, e-mail: Gianfranco.Piras@mail.wvu.edu
[2]Department of Economics, University of Maryland, College Park, MD 20742, Tel.: +1-301-405-3499, e-mail: prucha@econ.umd.edu

Abstract

This paper explores the properties of pre-test strategies in estimating a linear Cliff-Ord-type spatial model when the researcher is unsure about the nature of the spatial dependence. More specifically, the paper explores the finite sample properties of the pre-test estimators introduced in Florax *et al.* (2003), which are based on Lagrange Multiplier (LM) tests, within the context of a Monte Carlo study. The performance of those estimators is compared with that of the maximum likelihood (ML) estimator of the encompassing model. We find that, even in a very simple setting, the bias of the estimates generated by pre-testing strategies can be very large and the empirical size of tests can differ substantially from the nominal size. This is in contrast to the ML estimator. However, if the true data generating process corresponds to the spatial error or lag model the issues arising with the pre-test estimators seem to be lessened.

1 Introduction[1]

The recent literature on spatial econometrics has been concerned with model specification issues both in cross sectional as well as panel data analysis.[2] In the present paper we will focus exclusively on cross sectional models. Empirical work is often based on estimation strategies which entails estimating initially a linear model (i.e., without spatial dependencies), followed by testing for spatial dependences, and re-estimation if spatial dependence cannot be rejected. It is well known that, in general, pre-test strategies may potentially introduce bias for both the parameter estimates and corresponding standard errors.[3] Of course, on the other hand, efficiency may be lost when the researcher estimates a more general model than necessary. The purpose of this paper is to explore the implications of some common pre-test strategies used in the estimation of Cliff-Ord-type spatial models.

In estimating Cliff-Ord-type models two forms of spatial dependences are usually considered in applied work, which correspond to two different model specifications. The first form of spatial dependence relates to the error term and specifies a spatial auto regressive process for the disturbances. Correspondingly, the model that derives from it is often referred to as a spatial error model (see, e.g., Anselin, 1988b). The second form arises when the value of the dependent variable corresponding to each cross-sectional unit is jointly determined with the values at all other neighboring cross-sectional units. This is achieved through the inclusion of a weighted average of the dependent variable which is often described in the literature as a spatial lag. Consequently, the model that derives from it is referred to as a spatial autoregressive model or, simply, a spatial lag model (see, e.g., Anselin, 1988b).

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[2]See e.g., Cliff & Ord (1972, 1973, 1981); Florax & Folmer (1992); Anselin *et al.* (1996); Anselin (1988a); Florax *et al.* (2003); Baltagi *et al.* (2007, 2003); Baltagi & Liu (2008); Baltagi *et al.* (2009, 2008); Burridge (1980); Debarsy & Ertur (2010), among many others.

[3]See, e.g., Leeb & Poetscher (2008) for recent fundamental results and Judge *et al.* (1985, Ch. 3) for a classical text book presentation.

estimation of either the spatial lag or the spatial error models and never of a model that contains both error and lag dependences, i.e., of the encompassing or full model. A partial explanation for this may be that the parameters of the full model are not identified if the model does not contain exogenous variables. However, most empirical specifications include exogenous variables, and in this situation the parameters of the full model are identified under mild regularity conditions; see, e.g., Kelejian & Prucha (1998). Thus in this situation the researcher can estimate the parameters of the full model, and is not forced to select either the spatial lag or error model. Since the data may have been generated by the full model, which includes the spatial lag and error model as special cases, this reduces the likelihood of model misspecification. Of course, if the data have been generated by either the spatial lag or error model, estimating the full model will lead to a loss of efficiency. On the other hand, using a pre-testing strategy may yield biased estimates and may result in a situation where the employed asymptotic distribution of the estimator (derived without taking into account the pre-testing strategy) provides a bad approximation to the actual small sample distribution of the final stage estimator.

This paper explores the importance of the issues raised above for the estimation of linear Cliff-Ord-type spatial models. We explore within the context of a Monte Carlo study the small sample performance of the pre-test estimators which are described in Florax *et al.* (2003) and which are based on a series of LM tests. Florax *et al.* (2003) also report on a Monte Carlo study. Different from Florax *et al.* (2003) we also compare the performance of the pre-test estimators with that of the ML estimator of the encompassing model, which allows for spatial spill-overs in the endogenous variables and disturbances. Importantly, we also compare and report on the size of Wald-type tests associated with the pre-test estimators and the ML estimator of the encompassing model. Our results are cautionary, in that we find that even in simple settings the bias of the estimates generated by a pre-testing strategy can be very large and the empirical size of tests can differ substantially from the nominal size. Quite expectedly, the results also show that the ML estimator based on the full model is consistent, and the size of hypothesis tests is reasonably close to the nominal size. However, if the true data generating process corresponds to the spatial error or lag model the issues arising with the pre-test estimators seem to be lessened.

Section 2 briefly describes the models considered for this study and presents the corresponding likelihoods on which our estimators are based. Section 3 introduces the LM tests while Section 4 gives the pre-test estimators based on those LM tests. Section 5 describes the design of our Monte Carlo experiment and discusses the main evidence. Section 6 concludes and give indication for future work.

2 The models

As we mentioned in the introduction, much of the empirical spatial econometrics literature has focused on the estimation of two alternative models relating to different forms of spatial dependence. In one case, spatial dependence is introduced via the disturbance process, where the disturbance term corresponding to one location is assumed to be jointly determined with those at other locations. In the other case, the dependent variable at one location is assumed to be jointly determined by its values at other locations. From an empirical perspective, each of these two forms of dependence translates into a different Cliff-Ord-type spatial model. The model corresponding to the first case is known in the literature as spatial error model; while the model corresponding to the second case is known as the spatial lag model (Cliff & Ord, 1973; Anselin, 1988b). In what follows, we will briefly review these models and the corresponding likelihoods. Towards assessing the effects of pre-test strategies we will furthermore consider the encompassing Cliff-Ord-type spatial model, which includes both forms of spatial effects. As it is common in the literature, we refer to this model as a spatial autoregressive auto-regressive model (SARAR(1,1)); see e.g., Anselin (1988b). As remarked, the parameters of the SARAR(1,1) model can be consistently estimated under mild regularity conditions, provided the presence of exogenous variables; see, e.g., Kelejian & Prucha (1999, 2010); Florax & Folmer (1992); Anselin (1988b). Of course, if the true data generating process corresponds to a spatial error or lag model, we expect some loss in efficiency when estimating the encompassing SARAR(1,1) model. We will use the encompassing model to explore the properties of the considered pre-test estimators based on LM tests.

The approach frequently taken in empirical work is to start with the classical linear regression model

$$y = X\beta + \varepsilon, \tag{1}$$

where y is an $n \times 1$ vector of observations on the dependent variable, X is an $n \times k$ matrix of observations on the non-stochastic explanatory variables, β a $k \times 1$ vector of corresponding parameters, and ε an $n \times 1$ vector of innovations whose elements are - for simplicity - assumed in the following to be i.i.d. $N(0, \sigma^2)$.[4] Under regularity conditions the OLS estimator is also the ML estimator.

As an alternative to the linear regression model (1), the error term can be specified as a spatial autoregressive process, leading to the spatial error model

$$\begin{aligned} y =& X\beta + u, \\ u =& \rho W u + \varepsilon, \end{aligned} \tag{2}$$

where u is the $n \times 1$ vector of disturbances, W is an $n \times n$ non-stochastic weighting matrix,[5] ρ is a scalar spatial autoregressive parameter with $|\rho| < 1$, and all other variables are defined as above. For efficiency we can estimate the model in (2) by ML (Ord, 1975), although OLS remains unbiased. The expression for the log likelihood function of model (2) takes the form

$$L = -\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln(\sigma^2) + \ln|B| - \frac{1}{2\sigma^2}(y - X\beta)'B'B(y - X\beta), \tag{3}$$

where $B = I_N - \rho W$.[6] As an alternative to the ML estimator, a feasible GLS estimator, which utilizes a generalized method of moments estimator for ρ, has been suggested by Kelejian & Prucha (1999). However, in this paper we concentrate on the ML estimator.

A further alternative to the linear regression model (1) which is often estimated in the empirical literature is the spatial autoregressive model

$$y = \lambda W y + X\beta + \varepsilon, \tag{4}$$

where λ is a scalar spatial autoregressive parameter with $|\lambda| < 1$, and all other variables are defined as above. Because of the simultaneous nature of the spatial lag variable, Wy is correlated with the disturbance term ε. Thus

[4]We note that even for this standard setup our Monte Carlo results will (depending on the parameter constellations) detect sizable biases of the considered pre-test estimators.

[5]The assumptions made on the weights matrix are standard and we will not discuss them in this paper.

[6]For details on the maximum likelihood estimation see Anselin (1988b), Ch 12.

OLS is inconsistent, but the model can again be estimated efficiently by ML. The log likelihood takes the following form (Ord, 1975)

$$L = -\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln(\sigma^2) + \ln|A| - \frac{1}{2\sigma^2}(Ay - X\beta)'(Ay - X\beta), \quad (5)$$

where $A = I - \lambda W$. As an alternative to the ML estimator, the model could also be estimated by instrumental variables/generalized method of moments (Kelejian & Prucha, 1998), but again the present paper will focus on the ML estimator.

Finally, we consider the encompassing model which allows for spatial lags in the dependent variable, as well as in the disturbances, i.e.,

$$\begin{aligned} y &= \lambda W y + X\beta + u, \\ u &= \rho W u + \varepsilon. \end{aligned} \quad (6)$$

The log likelihood for this model is given by

$$\begin{aligned} L \quad = \quad & -\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln(\sigma^2) + \frac{n}{2}\ln|B| + \frac{n}{2}\ln|A| \\ & -\frac{1}{2\sigma^2}(Ay - X\beta)'B'B(Ay - X\beta). \end{aligned} \quad (7)$$

We emphasize that our terminology of referring to model (6) as the encompassing model should only be understood to apply "locally" in the sense that it encompasses models (1), (2) and (4). Of course, various further generalizations of model (6) have been considered in the literature. We purposefully focus our explorations on the above simple setup to avoid the contamination of the results from other modeling issues, e.g., the specification of the weights matrix.

3 LM tests

In this section we define the LM tests for the absence of spatial dependence employed in the construction of our considered pre-test estimators. Burridge (1980) derived the LM test statistic for the spatial error model. Anselin (1988b) derived LM tests for the more general SARAR model. Requiring only the estimation of the restricted specification, LM tests have been considered particularly appealing in a spatial setting because of the computational difficulties related to the maximum likelihood estimation of the spatial models.[7]

With the first of these tests we wish to evaluate the hypothesis that the disturbances are independently normally distributed with constant variance (i.e. $\rho = 0$) against the alternative that they are generated by the first order spatial autoregression in (2). The LM-test statistics for this hypothesis is given by

$$LM_\rho = \frac{[e'We/(e'e/n)]^2}{tr[W'W + WW]} \tag{8}$$

where $e = y - X\widehat{\beta}_{OLS}$ denotes the vector of OLS residuals, and tr is the trace operator.

The second LM-test statistics, which evaluates the null hypothesis that $\lambda = 0$ in (4) against the alternative of a spatial autoregressive process is given by

$$LM_\lambda = [e'Wy/(e'e/n)]^2/D \tag{9}$$

where $e = y - X\widehat{\beta}_{OLS}$ denotes, as before, the vector of OLS residuals, $D = [(WX\widehat{\beta}_{OLS})'M(WX\widehat{\beta}_{OLS})/\widehat{\sigma}^2_{OLS}]+tr(W'W+WW)$, $M = [I-X(X'X)^{-1}X']$, and $\widehat{\beta}_{OLS}$ and $\widehat{\sigma}^2_{OLS}$ are the OLS estimates of β and σ^2. Under the null hypothesis that the true model is (1) both LM_ρ and LM_λ are asymptotically distributed as $\chi^2(1)$.

The two test statistics presented above assume that the other form of dependence is not present. In other word, LM_ρ is derived under the null hypothesis that $\rho = 0$, but it assumes that also λ is zero. Using the general principles of specification testing with locally misspecified alternatives derived in Bera & Yoon (1993), Anselin *et al.* (1996) develop a set of diagnostics that are a robust version of (8) and (9). The expressions for the robust versions of the tests become, respectively,

$$LM^*_\rho = \frac{\{e'We/(e'e/n) - [tr(WW + W'W)/D]\, e'Wy/(e'e/n)\}^2}{tr(WW + W'W)[1 - tr(WW + W'W)/D]} \tag{10}$$

and

$$LM^*_\lambda = \frac{[e'Wy/(e'e/n) - e'We/(e'e/n)]^2}{[D - tr(WW + W'W)]} \tag{11}$$

[7] However, with the increase in power achieved by the modern computers, and the various methods to approximate the Jacobian term (LeSage & Pace, 2009), this problem has been somewhat mitigated.

where D as well as the other symbols where defined before. Both the LM^*_ρ and the LM^*_λ statistics are asymptotically distributed as $\chi^2(1)$.

4 Pre-test estimators

The pre-tests estimators are based on the sequence of the LM-tests presented in the previous section. We follow the algorithms illustrated in Florax *et al.* (2003), which propose three different "approaches" toward specification tests, each leading to a different pre-test estimator. Before reviewing these three approaches, we need to introduce some additional notation.

Let $\widehat{\beta}_{OLS}$ be the OLS estimator based on model (1), $\widehat{\rho}_{MLE}$ and $\widehat{\beta}_{MLE}$ the ML estimators corresponding to the model in (2) and, finally, $\widehat{\lambda}_{MLL}$ and $\widehat{\beta}_{MLL}$ the ML estimators corresponding to the model in (4). Also, let $\widehat{\lambda}_{ML}$, $\widehat{\rho}_{ML}$, and $\widehat{\beta}_{ML}$ denote the ML estimator for λ, ρ and β based on the full model in (6). Correspondingly, we have the following estimators for $\theta = (\lambda, \rho, \beta')'$:

$$\begin{aligned}
\widehat{\theta}_{OLS} &= (0, 0, \widehat{\beta}'_{OLS})', \\
\widehat{\theta}_{MLE} &= (0, \widehat{\rho}_{MLE}, \widehat{\beta}'_{MLE})', \\
\widehat{\theta}_{MLL} &= (\widehat{\lambda}_{MLL}, 0, \widehat{\beta}'_{MLL})', \\
\widehat{\theta}_{ML} &= (\widehat{\lambda}_{ML}, \widehat{\rho}_{ML}, \widehat{\beta}'_{ML})'.
\end{aligned}$$

The first approach in Florax *et al.* (2003) is based on the test statistics LM_ρ and LM_λ and can be summarized as follows:

1. Estimate the non-spatial model by OLS to obtain $\widehat{\beta}_{OLS}$ and the vector of OLS residuals $e = y - X\widehat{\beta}_{OLS}$.

2. Test the hypothesis of no spatial dependence due to an omitted spatially autoregressive error or to an omitted spatial lag using, respectively, LM_ρ and LM_λ.

3. If both tests statistics are not significant, then accept $H_0 : \lambda = \rho = 0$, and consequently the estimator for θ is given by $\widehat{\theta}_{OLS} = (0, 0, \widehat{\beta}'_{OLS})'$.

4. If LM_ρ is significant and LM_λ is not significant then accept $H^\rho_1 : \lambda = 0; \rho \neq 0$ and estimate model (2) by maximum likelihood to get $\widehat{\theta}_{MLE} = (0, \widehat{\rho}_{MLE}, \widehat{\beta}'_{MLE})'$.

5. If LM_λ is significant and LM_ρ is not significant then accept $H_1^\lambda : \lambda \neq 0; \rho = 0$ and estimate model (4) by maximum likelihood to get $\widehat{\theta}_{MLL} = (\widehat{\lambda}_{MLL}, 0, \widehat{\beta}'_{MLL})'$.

6. Finally, if both LM_λ and LM_ρ are significant, estimate the specification corresponding to the more significant of the two tests.

Florax *et al.* (2003) refer to this approach as the "classic" approach because it is based on the test statistics LM_ρ and LM_λ.

Let $\widehat{\theta}_{PT1}$ denote the estimator for θ based on this approach, where "PT" stands for "pre-test". Then this estimator is formally given by[8]

$$
\begin{aligned}
\widehat{\theta}_{PT1} &= (\widehat{\lambda}_{PT1}, \widehat{\rho}_{PT1}, \widehat{\beta}'_{PT1})' \\
&= \mathbf{1}\left(LM_\lambda < \chi_{.975}, LM_\rho < \chi_{.975}\right)\widehat{\theta}_{OLS} \\
&\quad +\mathbf{1}\left(LM_\lambda < \chi_{.975}, LM_\rho \geq \chi_{.975}\right)\widehat{\theta}_{MLE} \\
&\quad +\mathbf{1}\left(LM_\lambda \geq \chi_{.975}, LM_\rho < \chi_{.975}\right)\widehat{\theta}_{MLL} \\
&\quad +\mathbf{1}\left(LM_\lambda \geq \chi_{.975}, LM_\rho \geq \chi_{.975}\right)\mathbf{1}\left(LM_\lambda < LM_\rho\right)\widehat{\theta}_{MLE} \\
&\quad +\mathbf{1}\left(LM_\lambda \geq \chi_{.975}, LM_\rho \geq \chi_{.975}\right)\mathbf{1}\left(LM_\lambda \geq LM_\rho\right)\widehat{\theta}_{MLL}.
\end{aligned}
$$

where $\mathbf{1}(.)$ denotes the indicator function.

The second approach, that they refer to as the robust approach, is identical to the previous one with the exception that it is performed with the robust versions of the LM tests. Let $\widehat{\theta}_{PT2}$ denote the estimator for θ based on this approach, then it is formally given by

$$
\begin{aligned}
\widehat{\theta}_{PT2} &= (\widehat{\lambda}_{PT2}, \widehat{\rho}_{PT2}, \widehat{\beta}'_{PT2})' \\
&= \mathbf{1}\left(LM^*_\lambda < \chi_{.975}, LM^*_\rho < \chi_{.975}\right)\widehat{\theta}_{OLS} \\
&\quad +\mathbf{1}\left(LM^*_\lambda < \chi_{.975}, LM^*_\rho \geq \chi_{.975}\right)\widehat{\theta}_{MLE} \\
&\quad +\mathbf{1}\left(LM^*_\lambda \geq \chi_{.975}, LM^*_\rho < \chi_{.975}\right)\widehat{\theta}_{MLL} \\
&\quad +\mathbf{1}\left(LM^*_\lambda \geq \chi_{.975}, LM^*_\rho \geq \chi_{.975}\right)\mathbf{1}\left(LM^*_\lambda < LM^*_\rho\right)\widehat{\theta}_{MLE} \\
&\quad +\mathbf{1}\left(LM^*_\lambda \geq \chi_{.975}, LM^*_\rho \geq \chi_{.975}\right)\mathbf{1}\left(LM^*_\lambda \geq LM^*_\rho\right)\widehat{\theta}_{MLL}.
\end{aligned}
$$

[8] Since the number of tests carried out is two, Florax *et al.* (2003) suggest that the overall significance level of the strategy is the sum of the significance level of the two tests. This explains why the definition of $\widehat{\theta}_{PT1}$ is in terms of a $\chi_{.975}$.

Florax *et al.* (2003) consider a third approach which is a "hybrid" specification strategy in that it combines the use of both test statistics (classical and robust). It is identical to the classical approach, except that step 6 is modified as follows: If both LM_ρ and LM_λ are significant, estimate the specification pointed by the more significant of the two robust statistics LM_ρ^* and LM_λ^*. As pointed out by Florax *et al.* (2003) the performance of this hybrid pre-test estimator is identical to the classical pre-test estimator. The reason is that, analytically, $LM_\lambda \geq LM_\rho$ if and only if $LM_\lambda^* \geq LM_\rho^*$, as is easily checked.[9] We thus do not report separately on the performance of this hybrid pre-test estimator.

From the above definitions of the pre-test estimators it is obvious that they are highly nonlinear, and that they cannot generally be expected to be unbiased or consistent. Furthermore, it is obvious that the asymptotic distribution of those estimators will generally differ from those of $\widehat{\theta}_{OLS}$, $\widehat{\theta}_{MLL}$ and $\widehat{\theta}_{MLE}$. Of course, potential issues stemming from improper inference when using pre-test estimators are well known; see, e.g., Leeb & Poetscher (2008). The Monte Carlo study is intended to shed some light on the importance of those issues in the estimation of spatial models.

5 Monte Carlo

In what follows, we report on a Monte Carlo study of the small sample properties of the two pre-test estimators defined above. For comparison we also give results for the small sample properties of the ML estimator defined by the log-likelihood function of the comprehensive model (6). The design of the Monte Carlo is, intentionally, very simple.

Monte Carlo Design

In all of the experiments, the data are generated from the following simple model:

$$\begin{aligned} y &= \lambda W y + x_1 \beta_1 + x_2 \beta_2 + u, \\ u &= \rho W u + \varepsilon. \end{aligned} \tag{12}$$

Obviously, for $\lambda = 0$ or $\rho = 0$ this model includes, respectively, the spatial error or spatial lag model as special cases. The two regressors, x_1 and x_2, are

[9] Results on this are available from the authors.

normalized versions of income per capita and the proportion of housing units that are rental in 1980, in 760 counties in U.S. mid-western states.[10] We normalized the data by subtracting from each observation the corresponding sample average, and then dividing that result by the sample standard deviation. The first n values of these normalized variables were used in our Monte Carlo experiments. The regressors are treated as fixed and thus are held constant over all of the Monte Carlo trials. The Monte Carlo study assumes that units are located on a regular grid of dimension 23×23, which implies a sample size of $n = 529$. The spatial weighting matrix employed in our Monte Carlo study is based on the queen criteria (i.e. common borders and vertex). The values of β_1 and β_2 are set equal to 0.5. We consider the same set of values for both ρ and λ, namely -0.8,-0.6,-0.4,-0.2, 0, 0.2, 0.4, 0.6, 0.8. Finally, we assume that the elements of the innovation vector are i.i.d. $N(0, 1)$.[11] The results presented in the next section are based on 1,000 replications. All elaborations were performed using R statistical software (R Development Core Team, 2010) with the library **spdep** (Bivand *et al.* , 2010).[12]

Monte Carlo Results

The results of the Monte Carlo experiments are reported in Tables 1-6. Table 1 reports on the frequency with which the two considered pre-test procedures select the classical linear regression model (1), the spatial error model (2) and spatial lag model (4). This provides important background information towards interpreting the small sample properties of the pre-test estimators.

Tables 2-5 report on the biases and mean squared errors (MSEs) of the two pre-test estimators and the ML estimator of, respectively, ρ, λ, β_1, and β_2. In all tables, the first two columns contain the considered combinations of the true values of ρ and λ employed in generating the data. We note again

[10]These data were taken from Kelejian & Robinson (1995) and where also used by Arraiz *et al.* (2010).

[11]The target R^2 for the simulation was set to 0.3.

[12]We also run a Monte Carlo simulations with a higher $R^2 = 0.6$, and another simulations with a larger sample size (i.e., $n = 1,024$). The results are qualitatively similar to the ones reported here. For the higher R^2 we find for some parameter constellations somewhat smaller biases for the pre-test estimators, but we still find substantial size distortions for the corresponding tests. The additional Monte Carlo results are available from the authors upon request.

that if $\lambda = 0$ the data are generated from a spatial error model, and if $\rho = 0$ the data are generated from a spatial lag model. In columns three to five we report the biases of the respective estimators, and in columns six to eight the MSEs.

Finally, in Table 6 we report on the empirical size of tests of the null hypothesis that the parameters are equal to the true value corresponding to the pre-test procedures and the ML estimators, provided that the null hypothesis is true. All underlying "t-ratios" defining those tests are based on estimated standard errors, which are calculated from the negative inverse Hessian of the log-likelihood function. For the pre-test estimators – consistent with what seems to be the usual practice when such estimators are employed in empirical work – these calculations are based on the log-likelihood function corresponding to the model selected by the pre-test procedure. For the ML estimator the calculations are based on the log-likelihood function of the comprehensive model.

By definition the size of a test is the probability of falsely rejecting the null hypothesis. In case of pre-test estimators, the size is not only the probability of rejecting the null hypothesis when the true model has been selected, but the probability of selecting the wrong model should also be taken into account. Details of our size estimates are provided in the Appendix.

First consider the results in Table 1. One interesting finding in the tables is that for various parameter constellations where both ρ and λ are non-zero, the pretest procedures seem to end up selecting either the spatial lag or the spatial error model with very high frequency. For example, if $\rho = -0.8$ and $\lambda = -0.8$ the spatial lag model is selected in 95 percent of the cases. On the other hand, if $\rho = 0.8$ and $\lambda = 0.4$ the spatial error model is selected 97 percent of the cases. Recall again that the Monte Carlo study covers situations where the true data generating process corresponds to the spatial lag model or the spatial error model as special cases when $\rho = 0$ or $\lambda = 0$. An inspection of Table 1 shows that if the data are generated by the spatial lag model, i.e. $\rho = 0$, and $\lambda = 0.2$ or $\lambda = -0.2$ the frequency with which the pre-test procedures correctly selects the spatial lag model ranges only between 0.22 and 0.74. As a result, among other things, the pre-test estimates for λ will be seen to exhibit sizable bias for smaller values of λ. As the value of λ in modulus increases the frequency of selecting the spatial lag model increases towards one. Similarly, if the data are generated by the spatial error model, i.e. $\lambda = 0$, and $\rho = 0.2$ or $\rho = -0.2$ the frequency with which the pre-test procedures correctly selects the spatial error model ranges only between 0.13

and 0.54. As a result, among other things, the pre-test estimators for ρ will be seen to exhibit sizable biases for smaller values of ρ. As the value of ρ in modulus increases the frequency of selecting the spatial error model increases towards one.

In general, there seems to be a tendency of the two pre-test procedures to favor the estimation of the spatial lag model for negative values of ρ. For positive values of the error parameter the two pre-test procedures seem to favor the estimation of the spatial error model.

Next consider the results in Table 2. Looking at the averages, we note that the bias of $\widehat{\rho}_{ML}$ is considerably lower (0.0105) than that of the pre-test estimators $\widehat{\rho}_{PT1}$ and $\widehat{\rho}_{PT2}$ (0.2260 and 0.2345). The highest bias corresponds to situations where there is substantial spatial dependence both of the error and of the lag type. As an example, consider the first row when both ρ and λ are equal to -0.8. The bias for both pre-test estimators in this case equals 0.7254. This is because, as highlighted above, both pre-test procedures lead to the estimation of the spatial lag model in almost 95 percent of the cases for this combination of values. The bias is still very high when $\rho = -0.8$ and λ equals to -0.6, 0.6, and 0.8 because even in these cases the pre-test procedure tend to suggest overwhelmingly the estimation of a spatial lag model. The pre-test estimators also exhibit large biases for some of the cases where ρ is positive, but on average to a somewhat lesser degree than when ρ is negative. We note further that even if we focus on cases where the true data generating corresponds to the spatial error model, i.e., $\lambda = 0$, the pre-test estimators can still be, relative to the true parameter value, substantially biased - although to a lesser degree - for some of the cases considered. For example, for $\rho = 0.2$ and -0.2 the biases of the two pre-test estimators are -0.07 and 0.09, and -0.13 and 0.16, respectively.

In looking at the MSEs reported in Table 2 we see that the MSEs of the two pre-test estimators for ρ are very similar, while (on average) the MSE of the ML estimator is considerably lower (0.1361 vs. 0.0160).

Table 3 displays the bias and MSE for the estimators of λ. Consider first the results in terms of bias. Looking at the averages, the results for the estimators of λ are in line with what was found for the estimators of ρ. More specifically, the average bias for $\widehat{\lambda}_{ML}$ is considerably lower (0.0086) than those of the pre-test estimators $\widehat{\lambda}_{PT1}$ and $\widehat{\lambda}_{PT2}$ (0.1735 and 0.1743). An inspection of the table also shows that some of the biases exhibited by the pre-test estimators are very high. For example for $\rho = 0.8$ and $\lambda = -.8$

the bias is 0.8 for both pre-test estimator. We note further that even if we focus on cases where the true data generating corresponds to the spatial lag model, i.e., $\rho = 0$, the pre-test estimators can still be, relative to the true parameter value, substantially biased - although to a lesser degree - for some of the cases considered. For example, for $\lambda = 0.2$ and -0.2 the biases of the two pre-test estimators are -0.04 and 0.08, and -0.09 and 0.11, respectively.

Results for the MSE in Tables 3 are also in line with the findings for the estimators for ρ, with the MSE of $\widehat{\lambda}_{ML}$ being on average substantially lower (0.0125) to those of either $\widehat{\lambda}_{PT1}$ and $\widehat{\lambda}_{PT2}$ (0.0900 and 0.0896).

The results in Tables 4 and 5 pertain to the bias and MSE of the estimators of β_1 and β_2, respectively. Since the results for the two parameters are very similar, we will just focus on the results for β_1. On average, the bias of both $\widehat{\beta}_{1,PT1}$ and $\widehat{\beta}_{1,PT2}$ is larger than that of $\widehat{\beta}_{1,ML}$. For some parameter constellations the bias of the pre-test procedures can be sizable. In particular, for $\rho = -0.8$ and $\lambda = 0.4$ the bias of $\widehat{\beta}_{1,PT1}$ and $\widehat{\beta}_{1,PT2}$ are 0.1049 and 0.1096, and that of $\widehat{\beta}_{1,ML}$ is 0.0001. However, when ρ or λ is equal zero $\widehat{\beta}_{1,PT1}$ and $\widehat{\beta}_{1,PT2}$ exhibit only small biases and perform very similarly to the $\widehat{\beta}_{1,ML}$.

We next discuss the results reported in Table 6. As discussed above, this table display the results on the empirical size of a Wald test of the hypothesis that a respective parameter is equal to the true parameter value, where the intended size is 5%. Not surprisingly, in light of the sizable biases reported in Tables 2-5 the tests corresponding to the pre-test estimators can exhibit large distortions in size, depending on parameter constellations. On average the size of the tests corresponding to the pre-test estimators for λ and ρ exceeds 70%, and those corresponding to the pre-test estimators for β_1 and β_2 exceed 12% and 7%. In contrast, the empirical size of the tests bases on the ML estimator from the comprehensive model is found to be on average much closer to the nominal size of the test of 5%. The size distortions of the tests associated with the pre-test estimators can be large even for cases where the true data generating process is either the spatial lag model or the spatial error model; for example, see the results for $\rho = 0$ and $\lambda = 0.2$ or -0.2, and the results for $\lambda = 0$ and $\rho = 0.2$ or $\rho = -0.2$. The tables reveal cases where the size is too large as well as cases where the size is too small.

The results reported in Tables 2-6 suggest that, even for very simple spatial data generating processes as those underlying our Monte Carlo analysis, the bias of the pre-test estimators can be very large in many cases. Also the size of the test for the pre-test estimators can be far away from the 5%

nominal value. On the other hand, the ML estimators based on the full model are consistent for all the parameter values. Furthermore, the size of the tests associated with the ML estimator is only significantly different from the nominal value for very large values of the spatial parameters.[13] However, if the true data generating process corresponds to the spatial error or lag model the issues arising with the pre-test estimators seem to be lessened.

6 Conclusions

This paper examined the small sample properties of model selection and pre-test procedures in spatial econometrics. In particular, we consider a cross-sectional Cliff-Ord-type model, and examine the small sample properties of pre-test estimators suggested in Florax *et al.* (2003). We also explore the properties of corresponding Wald tests. For comparison we also report on the small sample properties of the ML estimator for the comprehensive model, and corresponding Wald tests.

Our Monte Carlo design is purposefully kept simple to avoid the contamination of the results from other modeling issues. For the same reason we draw the innovations from a Gaussian distribution, and we use the ML approach based on a Gaussian likelihood in the estimation of sub models and of the comprehensive model. The tests employed in the pre-test are likelihood ratio tests based on a Gaussian likelihood. Even within our simple setup we find that the biases of the estimators generated by the pre-testing strategies can be large and the size of hypothesis test may be quite different from the envisioned nominal size. In contrast, and not surprising, the ML estimator based on the comprehensive model is consistent, and the size of hypothesis tests is reasonably close to the nominal values. We also find that the loss in efficiency of estimating the full model, when the data are actually generated by a sub model, is modest.

We understand that most empirical work involves a certain amount of model selection and pre-testing. However, our results provide a caution in that they suggest that for the estimation of a cross-sectional Cliff-Ord-type model pre-test strategies may be associated with substantial pitfalls. The parameters of the comprehensive model are at this point well understood to be identified and consistently estimable under fairly general assumptions,

[13]However, this is simply a small sample problem. In fact, considering a larger sample size ($n = 1,029$) the results for the size came out much closer to 5%.

given some of the regressors are exogenous.[14] Our results pertaining to the estimation of the comprehensive model are encouraging, and provide the empirical researchers with a practical and robust option, which avoids problems arising from the pre-test strategies. The results also suggest that the potential efficiency losses from estimating the comprehensive model (when ρ or λ are zero) can be modest. We also not that if the true data generating process corresponds to the spatial error or lag model the issues arising with the pre-test estimators seem to be lessened.

Recently, Debarsy & Ertur (2010) suggested an extension of the pre-test estimators considered by Florax *et al.* (2003) to a panel framework. In future research it may be of interest to explore the properties of those extensions to a panel data setting.

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[14]Of course, as has been pointed out in the literature, without exogenous regressors the parameters are not identified.

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Table 1: Frequency of Selecting OLS, Spatial Error and Spatial Lag Model with Pre-test Procedures

		Prestest Estimator 1			Prestest Estimator 2		
ρ	λ	OLS	Error Model	Lag Model	OLS	Error Model	Lag Model
−0.8	−0.8	0.000	0.052	0.948	0.000	0.052	0.948
−0.8	−0.6	0.000	0.187	0.813	0.000	0.187	0.813
−0.8	−0.4	0.000	0.515	0.485	0.011	0.508	0.481
−0.8	−0.2	0.000	0.878	0.122	0.022	0.866	0.112
−0.8	0	0.000	0.992	0.008	0.022	0.975	0.003
−0.8	0.2	0.000	1.000	0.000	0.001	0.999	0.000
−0.8	0.4	0.334	0.585	0.081	0.001	0.807	0.192
−0.8	0.6	0.000	0.000	1.000	0.000	0.000	1.000
−0.8	0.8	0.000	0.000	1.000	0.000	0.000	1.000
−0.6	−0.8	0.000	0.024	0.976	0.003	0.023	0.974
−0.6	−0.6	0.000	0.096	0.904	0.005	0.094	0.901
−0.6	−0.4	0.000	0.345	0.655	0.074	0.297	0.629
−0.6	−0.2	0.000	0.751	0.249	0.216	0.619	0.165
−0.6	0	0.000	0.970	0.030	0.134	0.862	0.004
−0.6	0.2	0.038	0.962	0.000	0.047	0.953	0.000
−0.6	0.4	0.441	0.027	0.532	0.002	0.190	0.808
−0.6	0.6	0.000	0.000	1.000	0.000	0.000	1.000
−0.6	0.8	0.000	0.000	1.000	0.000	0.000	1.000
−0.4	−0.8	0.000	0.004	0.996	0.001	0.004	0.995
−0.4	−0.6	0.000	0.043	0.957	0.027	0.031	0.942
−0.4	−0.4	0.000	0.185	0.815	0.190	0.107	0.703
−0.4	−0.2	0.000	0.549	0.451	0.532	0.275	0.193
−0.4	0	0.014	0.905	0.081	0.496	0.500	0.004
−0.4	0.2	0.684	0.310	0.006	0.308	0.593	0.099
−0.4	0.4	0.049	0.000	0.951	0.009	0.002	0.989
−0.4	0.6	0.000	0.000	1.000	0.000	0.000	1.000
−0.4	0.8	0.000	0.000	1.000	0.000	0.000	1.000
−0.2	−0.8	0.000	0.001	0.999	0.000	0.001	0.999
−0.2	−0.6	0.000	0.014	0.986	0.027	0.003	0.970
−0.2	−0.4	0.000	0.095	0.905	0.255	0.030	0.715
−0.2	−0.2	0.001	0.374	0.625	0.727	0.070	0.203
−0.2	0	0.506	0.400	0.094	0.858	0.133	0.009
−0.2	0.2	0.779	0.003	0.218	0.610	0.040	0.350
−0.2	0.4	0.002	0.001	0.997	0.026	0.000	0.974
−0.2	0.6	0.000	0.001	0.999	0.000	0.001	0.999
−0.2	0.8	0.000	0.000	1.000	0.000	0.000	1.000
0	−0.8	0.000	0.000	1.000	0.001	0.000	0.999
0	−0.6	0.000	0.003	0.997	0.029	0.000	0.971
0	−0.4	0.001	0.027	0.972	0.248	0.001	0.751
0	−0.2	0.321	0.123	0.556	0.773	0.004	0.223
0	0	0.957	0.017	0.026	0.960	0.018	0.022
0	0.2	0.128	0.148	0.724	0.583	0.018	0.399
0	0.4	0.000	0.025	0.975	0.017	0.021	0.962
0	0.6	0.000	0.005	0.995	0.000	0.005	0.995
0	0.8	0.000	0.000	1.000	0.000	0.000	1.000
0.2	−0.8	0.000	0.000	1.000	0.000	0.000	1.000
0.2	−0.6	0.000	0.001	0.999	0.029	0.000	0.971
0.2	−0.4	0.169	0.007	0.824	0.258	0.002	0.740
0.2	−0.2	0.919	0.035	0.046	0.705	0.127	0.168
0.2	0	0.316	0.542	0.142	0.700	0.272	0.028
0.2	0.2	0.004	0.465	0.531	0.226	0.358	0.416
0.2	0.4	0.000	0.211	0.789	0.000	0.211	0.789
0.2	0.6	0.000	0.060	0.940	0.000	0.060	0.940
0.2	0.8	0.000	0.005	0.995	0.000	0.005	0.995
0.4	−0.8	0.000	0.000	1.000	0.000	0.000	1.000
0.4	−0.6	0.194	0.002	0.804	0.044	0.014	0.942
0.4	−0.4	0.798	0.134	0.068	0.203	0.447	0.350
0.4	−0.2	0.175	0.813	0.012	0.232	0.765	0.003
0.4	0	0.002	0.913	0.085	0.130	0.826	0.044
0.4	0.2	0.000	0.787	0.213	0.005	0.785	0.210
0.4	0.4	0.000	0.598	0.402	0.000	0.598	0.402
0.4	0.6	0.000	0.300	0.700	0.000	0.300	0.700
0.4	0.8	0.000	0.049	0.951	0.000	0.049	0.951
0.6	−0.8	0.376	0.062	0.562	0.009	0.163	0.828
0.6	−0.6	0.369	0.619	0.012	0.032	0.865	0.103
0.6	−0.4	0.021	0.977	0.002	0.024	0.973	0.003
0.6	−0.2	0.000	0.996	0.004	0.007	0.992	0.001
0.6	0	0.000	0.990	0.010	0.000	0.990	0.010
0.6	0.2	0.000	0.954	0.046	0.000	0.954	0.046
0.6	0.4	0.000	0.866	0.134	0.000	0.866	0.134
0.6	0.6	0.000	0.643	0.357	0.000	0.643	0.357
0.6	0.8	0.000	0.222	0.778	0.000	0.222	0.778
0.8	−0.8	0.007	0.992	0.001	0.001	0.998	0.001
0.8	−0.6	0.000	0.999	0.001	0.000	0.999	0.001
0.8	−0.4	0.000	1.000	0.000	0.000	1.000	0.000
0.8	−0.2	0.000	0.999	0.001	0.000	0.999	0.001
0.8	0	0.000	0.995	0.005	0.000	0.995	0.005
0.8	0.2	0.000	0.974	0.026	0.000	0.974	0.026
0.8	0.4	0.000	0.932	0.068	0.000	0.932	0.068
0.8	0.6	0.000	0.767	0.233	0.000	0.767	0.233
0.8	0.8	0.000	0.396	0.604	0.000	0.396	0.604

Table 2: Bias and MSE of $\widehat{\rho}_{PT1}$, $\widehat{\rho}_{PT2}$, and $\widehat{\rho}_{ML}$.

		BIAS			MSE		
ρ	λ	$\widehat{\rho}_{PT1}$	$\widehat{\rho}_{PT2}$	$\widehat{\rho}_{ML}$	$\widehat{\rho}_{PT1}$	$\widehat{\rho}_{PT2}$	$\widehat{\rho}_{ML}$
-0.8	-0.8	0.7254	0.7254	0.0083	0.6278	0.6278	0.0263
-0.8	-0.6	0.5576	0.5576	0.0011	0.5673	0.5673	0.0231
-0.8	-0.4	0.2043	0.2111	-0.0006	0.3790	0.3832	0.0212
-0.8	-0.2	-0.0726	-0.0621	-0.0017	0.1169	0.1245	0.0186
-0.8	0	-0.0027	0.0095	-0.0026	0.0129	0.0235	0.0166
-0.8	0.2	0.2225	0.2230	0.0002	0.0583	0.0588	0.0139
-0.8	0.4	0.6128	0.5710	-0.0091	0.4037	0.3449	0.0142
-0.8	0.6	0.8000	0.8000	0.0064	0.6400	0.6400	0.0115
-0.8	0.8	0.8000	0.8000	-0.0054	0.6400	0.6400	0.0098
-0.6	-0.8	0.5682	0.5695	-0.0183	0.3640	0.3639	0.0271
-0.6	-0.6	0.4886	0.4909	-0.0127	0.3560	0.3561	0.0242
-0.6	-0.4	0.2574	0.3018	0.0029	0.2912	0.3032	0.0236
-0.6	-0.2	-0.0076	0.0908	0.0035	0.1282	0.1722	0.0205
-0.6	0	0.0039	0.0628	-0.0049	0.0191	0.0569	0.0193
-0.6	0.2	0.2335	0.2395	-0.0015	0.0670	0.0716	0.0171
-0.6	0.4	0.5931	0.5677	0.0045	0.3535	0.3272	0.0144
-0.6	0.6	0.6000	0.6000	-0.0011	0.3600	0.3600	0.0121
-0.6	0.8	0.6000	0.6000	-0.0001	0.3600	0.3600	0.0104
-0.4	-0.8	0.3953	0.3953	-0.0141	0.1618	0.1618	0.0242
-0.4	-0.6	0.3569	0.3682	-0.0107	0.1690	0.1673	0.0251
-0.4	-0.4	0.2467	0.3077	-0.0028	0.1659	0.1666	0.0237
-0.4	-0.2	0.0600	0.2199	0.0010	0.1030	0.1359	0.0209
-0.4	0	0.0225	0.1752	-0.0083	0.0227	0.0849	0.0213
-0.4	0.2	0.3162	0.2752	-0.0024	0.1164	0.0897	0.0176
-0.4	0.4	0.4000	0.3997	-0.0075	0.1600	0.1598	0.0163
-0.4	0.6	0.4000	0.4000	0.0028	0.1600	0.1600	0.0128
-0.4	0.8	0.4000	0.4000	-0.0034	0.1600	0.1600	0.0112
-0.2	-0.8	0.1991	0.1991	-0.0140	0.0405	0.0405	0.0242
-0.2	-0.6	0.1885	0.1976	-0.0171	0.0449	0.0410	0.0224
-0.2	-0.4	0.1396	0.1798	-0.0142	0.0548	0.0457	0.0227
-0.2	-0.2	0.0371	0.1650	-0.0113	0.0489	0.0439	0.0229
-0.2	0	0.0859	0.1605	-0.0119	0.0283	0.0367	0.0203
-0.2	0.2	0.1997	0.1947	0.0008	0.0401	0.0387	0.0185
-0.2	0.4	0.2002	0.2000	-0.0053	0.0401	0.0400	0.0171
-0.2	0.6	0.2006	0.2006	-0.0002	0.0406	0.0406	0.0144
-0.2	0.8	0.2000	0.2000	-0.0055	0.0400	0.0400	0.0114
0	-0.8	0.0000	0.0000	-0.0268	0.0000	0.0000	0.0218
0	-0.6	-0.0020	0.0000	-0.0176	0.0014	0.0000	0.0210
0	-0.4	-0.0118	-0.0005	-0.0130	0.0053	0.0003	0.0217
0	-0.2	-0.0358	-0.0013	-0.0137	0.0107	0.0004	0.0216
0	0	-0.0002	-0.0003	-0.0130	0.0009	0.0005	0.0206
0	0.2	0.0375	0.0058	-0.0083	0.0100	0.0019	0.0195
0	0.4	0.0113	0.0099	-0.0054	0.0053	0.0047	0.0170
0	0.6	0.0034	0.0034	0.0023	0.0024	0.0024	0.0148
0	0.8	0.0000	0.0000	-0.0042	0.0000	0.0000	0.0115
0.2	-0.8	-0.2000	-0.2000	-0.0143	0.0400	0.0400	0.0173
0.2	-0.6	-0.2005	-0.2000	-0.0186	0.0404	0.0400	0.0188
0.2	-0.4	-0.2022	-0.1998	-0.0164	0.0416	0.0399	0.0188
0.2	-0.2	-0.1941	-0.1831	-0.0133	0.0390	0.0357	0.0195
0.2	0	-0.0681	-0.1302	-0.0056	0.0207	0.0309	0.0190
0.2	0.2	-0.0170	-0.0527	-0.0067	0.0407	0.0428	0.0199
0.2	0.4	-0.0791	-0.0791	-0.0129	0.0615	0.0615	0.0174
0.2	0.6	-0.1547	-0.1547	0.0002	0.0561	0.0561	0.0153
0.2	0.8	-0.1955	-0.1955	-0.0016	0.0422	0.0422	0.0112
0.4	-0.8	-0.4000	-0.4000	-0.0227	0.1600	0.1600	0.0119
0.4	-0.6	-0.3996	-0.3984	-0.0170	0.1598	0.1589	0.0138
0.4	-0.4	-0.3713	-0.3338	-0.0155	0.1434	0.1181	0.0151
0.4	-0.2	-0.1864	-0.2022	-0.0152	0.0478	0.0562	0.0152
0.4	0	-0.0396	-0.0665	-0.0245	0.0179	0.0309	0.0180
0.4	0.2	0.0364	0.0356	-0.0157	0.0550	0.0553	0.0178
0.4	0.4	0.0176	0.0176	-0.0053	0.1186	0.1186	0.0161
0.4	0.6	-0.1517	-0.1517	-0.0025	0.1671	0.1671	0.0152
0.4	0.8	-0.3544	-0.3544	0.0056	0.1659	0.1659	0.0128
0.6	-0.8	-0.5872	-0.5737	-0.0137	0.3474	0.3330	0.0070
0.6	-0.6	-0.4465	-0.4134	-0.0145	0.2157	0.1806	0.0089
0.6	-0.4	-0.2706	-0.2728	-0.0271	0.0805	0.0825	0.0112
0.6	-0.2	-0.1286	-0.1299	-0.0184	0.0212	0.0224	0.0110
0.6	0	-0.0076	-0.0076	-0.0147	0.0065	0.0065	0.0123
0.6	0.2	0.0775	0.0775	-0.0188	0.0298	0.0298	0.0134
0.6	0.4	0.1041	0.1041	-0.0131	0.0884	0.0884	0.0140
0.6	0.6	-0.0224	-0.0224	-0.0035	0.1860	0.1860	0.0135
0.6	0.8	-0.3860	-0.3860	0.0108	0.3095	0.3095	0.0131
0.8	-0.8	-0.3991	-0.3983	-0.0127	0.1673	0.1661	0.0030
0.8	-0.6	-0.2760	-0.2760	-0.0128	0.0809	0.0809	0.0033
0.8	-0.4	-0.1702	-0.1702	-0.0095	0.0318	0.0318	0.0034
0.8	-0.2	-0.0848	-0.0848	-0.0194	0.0097	0.0097	0.0050
0.8	0	-0.0094	-0.0094	-0.0232	0.0044	0.0044	0.0065
0.8	0.2	0.0411	0.0411	-0.0246	0.0212	0.0212	0.0089
0.8	0.4	0.0556	0.0556	-0.0336	0.0568	0.0568	0.0110
0.8	0.6	-0.0626	-0.0626	-0.0293	0.1692	0.1692	0.0104
0.8	0.8	-0.4083	-0.4083	-0.0122	0.4007	0.4007	0.0088
average		0.2260	0.2345	0.0105	0.1361	0.1363	0.0160

Table 3: Bias and MSE of $\widehat{\lambda}_{PT1}$, $\widehat{\lambda}_{PT2}$, and $\widehat{\lambda}_{ML}$.

		BIAS			MSE		
ρ	λ	$\widehat{\lambda}_{PT1}$	$\widehat{\lambda}_{PT2}$	$\widehat{\lambda}_{ML}$	$\widehat{\lambda}_{PT1}$	$\widehat{\lambda}_{PT2}$	$\widehat{\lambda}_{ML}$
-0.8	-0.8	-0.4214	-0.4214	-0.0061	0.2627	0.2627	0.0186
-0.8	-0.6	-0.3123	-0.3123	-0.0012	0.2923	0.2923	0.0162
-0.8	-0.4	-0.0612	-0.0577	-0.0048	0.2318	0.2315	0.0143
-0.8	-0.2	0.1081	0.1147	-0.0021	0.0731	0.0713	0.0107
-0.8	0	-0.0048	-0.0020	-0.0072	0.0029	0.0013	0.0080
-0.8	0.2	-0.2000	-0.2000	-0.0082	0.0400	0.0400	0.0054
-0.8	0.4	-0.3882	-0.3770	-0.0034	0.1523	0.1445	0.0035
-0.8	0.6	-0.2585	-0.2585	-0.0072	0.0695	0.0695	0.0019
-0.8	0.8	-0.1276	-0.1276	-0.0030	0.0172	0.0172	0.0005
-0.6	-0.8	-0.3442	-0.3420	0.0125	0.1544	0.1555	0.0182
-0.6	-0.6	-0.3034	-0.3005	0.0060	0.1827	0.1833	0.0164
-0.6	-0.4	-0.1351	-0.1159	-0.0076	0.1722	0.1733	0.0148
-0.6	-0.2	0.0442	0.0934	-0.0076	0.0763	0.0669	0.0121
-0.6	0	-0.0126	-0.0018	-0.0098	0.0054	0.0008	0.0095
-0.6	0.2	-0.2000	-0.2000	-0.0102	0.0400	0.0400	0.0065
-0.6	0.4	-0.3073	-0.2799	-0.0091	0.1027	0.0835	0.0038
-0.6	0.6	-0.1910	-0.1910	-0.0045	0.0386	0.0386	0.0018
-0.6	0.8	-0.0965	-0.0965	-0.0040	0.0101	0.0101	0.0005
-0.4	-0.8	-0.2517	-0.2507	0.0098	0.0723	0.0729	0.0164
-0.4	-0.6	-0.2339	-0.2225	0.0050	0.0909	0.0959	0.0169
-0.4	-0.4	-0.1577	-0.0875	-0.0067	0.0996	0.1113	0.0149
-0.4	-0.2	-0.0228	0.0977	-0.0075	0.0632	0.0541	0.0127
-0.4	0	-0.0252	-0.0015	-0.0062	0.0082	0.0006	0.0102
-0.4	0.2	-0.1992	-0.1911	-0.0052	0.0398	0.0373	0.0072
-0.4	0.4	-0.1868	-0.1831	-0.0052	0.0397	0.0371	0.0043
-0.4	0.6	-0.1311	-0.1311	-0.0076	0.0191	0.0191	0.0021
-0.4	0.8	-0.0653	-0.0653	-0.0025	0.0051	0.0051	0.0007
-0.2	-0.8	-0.1313	-0.1313	0.0105	0.0231	0.0231	0.0168
-0.2	-0.6	-0.1294	-0.1189	0.0103	0.0299	0.0355	0.0163
-0.2	-0.4	-0.0964	-0.0015	0.0034	0.0400	0.0678	0.0167
-0.2	-0.2	-0.0232	0.1206	-0.0017	0.0334	0.0403	0.0143
-0.2	0	-0.0212	-0.0016	0.0012	0.0050	0.0005	0.0107
-0.2	0.2	-0.1648	-0.1545	-0.0093	0.0317	0.0283	0.0081
-0.2	0.4	-0.0953	-0.1006	-0.0066	0.0123	0.0152	0.0058
-0.2	0.6	-0.0691	-0.0691	-0.0068	0.0069	0.0069	0.0026
-0.2	0.8	-0.0368	-0.0368	-0.0043	0.0020	0.0020	0.0008
0	-0.8	-0.0004	0.0003	0.0163	0.0059	0.0066	0.0160
0	-0.6	0.0005	0.0135	0.0084	0.0067	0.0157	0.0169
0	-0.4	0.0074	0.0868	0.0028	0.0097	0.0438	0.0165
0	-0.2	0.0626	0.1424	0.0011	0.0205	0.0327	0.0166
0	0	-0.0012	-0.0003	-0.0067	0.0008	0.0004	0.0138
0	0.2	-0.0433	-0.1075	-0.0037	0.0128	0.0253	0.0101
0	0.4	-0.0154	-0.0193	-0.0094	0.0068	0.0087	0.0068
0	0.6	-0.0076	-0.0076	-0.0101	0.0035	0.0035	0.0038
0	0.8	-0.0039	-0.0039	-0.0057	0.0006	0.0006	0.0012
0.2	-0.8	0.1551	0.1551	0.0072	0.0301	0.0301	0.0156
0.2	-0.6	0.1569	0.1672	0.0072	0.0311	0.0395	0.0174
0.2	-0.4	0.1753	0.2014	-0.0009	0.0444	0.0579	0.0188
0.2	-0.2	0.1907	0.1784	0.0014	0.0382	0.0345	0.0180
0.2	0	0.0272	0.0052	-0.0099	0.0054	0.0012	0.0163
0.2	0.2	-0.0259	-0.0574	-0.0089	0.0290	0.0328	0.0143
0.2	0.4	-0.0067	-0.0067	-0.0054	0.0430	0.0430	0.0095
0.2	0.6	0.0287	0.0287	-0.0102	0.0272	0.0272	0.0053
0.2	0.8	0.0307	0.0307	-0.0064	0.0049	0.0049	0.0015
0.4	-0.8	0.3467	0.3467	0.0158	0.1273	0.1273	0.0150
0.4	-0.6	0.3824	0.3655	0.0129	0.1607	0.1423	0.0185
0.4	-0.4	0.3857	0.3527	0.0010	0.1516	0.1293	0.0207
0.4	-0.2	0.2024	0.2001	-0.0005	0.0414	0.0401	0.0194
0.4	0	0.0259	0.0146	0.0049	0.0081	0.0049	0.0194
0.4	0.2	-0.1030	-0.1040	-0.0029	0.0459	0.0460	0.0171
0.4	0.4	-0.1563	-0.1563	-0.0148	0.1135	0.1135	0.0137
0.4	0.6	-0.0768	-0.0768	-0.0157	0.1239	0.1239	0.0095
0.4	0.8	0.0342	0.0342	-0.0182	0.0374	0.0374	0.0051
0.6	-0.8	0.6630	0.6277	0.0152	0.4561	0.4045	0.0145
0.6	-0.6	0.5976	0.5867	0.0092	0.3576	0.3459	0.0183
0.6	-0.4	0.4005	0.4001	0.0197	0.1606	0.1602	0.0208
0.6	-0.2	0.2012	0.2004	0.0091	0.0409	0.0403	0.0217
0.6	0	0.0047	0.0047	0.0015	0.0022	0.0022	0.0223
0.6	0.2	-0.1719	-0.1719	-0.0003	0.0460	0.0460	0.0206
0.6	0.4	-0.3035	-0.3035	-0.0126	0.1525	0.1525	0.0176
0.6	0.6	-0.3024	-0.3024	-0.0231	0.2512	0.2512	0.0136
0.6	0.8	-0.0775	-0.0775	-0.0300	0.1552	0.1552	0.0102
0.8	-0.8	0.8003	0.7998	0.0167	0.6405	0.6398	0.0126
0.8	-0.6	0.6003	0.6003	0.0163	0.3605	0.3605	0.0149
0.8	-0.4	0.4000	0.4000	0.0066	0.1600	0.1600	0.0165
0.8	-0.2	0.2006	0.2006	0.0205	0.0406	0.0406	0.0206
0.8	0	0.0034	0.0034	0.0221	0.0023	0.0023	0.0223
0.8	0.2	-0.1794	-0.1794	0.0175	0.0481	0.0481	0.0226
0.8	0.4	-0.3418	-0.3418	0.0180	0.1633	0.1633	0.0215
0.8	0.6	-0.3841	-0.3841	0.0078	0.3010	0.3010	0.0144
0.8	0.8	-0.2104	-0.2104	-0.0215	0.2722	0.2722	0.0100
average		0.1735	0.1743	0.0086	0.0900	0.0896	0.0125

Table 4: Bias and MSE of $\widehat{\beta}_{1,PT1}$, $\widehat{\beta}_{1,PT2}$, and $\widehat{\beta}_{1,ML}$.

		BIAS			MSE		
ρ	λ	$\widehat{\beta}_{1,PT1}$	$\widehat{\beta}_{1,PT2}$	$\widehat{\beta}_{1,ML}$	$\widehat{\beta}_{1,PT1}$	$\widehat{\beta}_{1,PT2}$	$\widehat{\beta}_{1,ML}$
−0.8	−0.8	-0.0015	-0.0015	-0.0021	0.0033	0.0033	0.0018
−0.8	−0.6	-0.0083	-0.0083	-0.0009	0.0051	0.0051	0.0018
−0.8	−0.4	-0.0331	-0.0329	-0.0019	0.0069	0.0068	0.0019
−0.8	−0.2	-0.0405	-0.0404	-0.0022	0.0043	0.0043	0.0019
−0.8	0	0.0007	0.0010	0.0008	0.0014	0.0014	0.0019
−0.8	0.2	0.0642	0.0643	0.0038	0.0059	0.0059	0.0020
−0.8	0.4	0.1049	0.1096	0.0001	0.0142	0.0151	0.0022
−0.8	0.6	0.0780	0.0780	0.0023	0.0085	0.0085	0.0020
−0.8	0.8	0.0765	0.0765	0.0044	0.0088	0.0088	0.0022
−0.6	−0.8	0.0023	0.0023	-0.0013	0.0024	0.0024	0.0018
−0.6	−0.6	0.0000	-0.0001	-0.0028	0.0033	0.0033	0.0018
−0.6	−0.4	-0.0132	-0.0119	-0.0004	0.0048	0.0045	0.0019
−0.6	−0.2	-0.0246	-0.0231	0.0004	0.0037	0.0034	0.0021
−0.6	0	0.0014	0.0027	0.0005	0.0018	0.0018	0.0022
−0.6	0.2	0.0517	0.0520	0.0025	0.0044	0.0044	0.0020
−0.6	0.4	0.0633	0.0614	0.0034	0.0066	0.0065	0.0023
−0.6	0.6	0.0583	0.0583	0.0012	0.0058	0.0058	0.0022
−0.6	0.8	0.0607	0.0607	0.0065	0.0062	0.0062	0.0021
−0.4	−0.8	0.0015	0.0014	-0.0032	0.0019	0.0019	0.0018
−0.4	−0.6	0.0038	0.0038	-0.0016	0.0024	0.0023	0.0019
−0.4	−0.4	0.0005	-0.0008	0.0003	0.0031	0.0029	0.0020
−0.4	−0.2	-0.0113	-0.0141	-0.0018	0.0028	0.0025	0.0019
−0.4	0	0.0017	0.0023	-0.0022	0.0018	0.0018	0.0021
−0.4	0.2	0.0336	0.0364	0.0004	0.0033	0.0035	0.0022
−0.4	0.4	0.0367	0.0361	0.0003	0.0034	0.0034	0.0021
−0.4	0.6	0.0379	0.0379	0.0005	0.0037	0.0037	0.0022
−0.4	0.8	0.0375	0.0375	0.0008	0.0039	0.0039	0.0023
−0.2	−0.8	0.0038	0.0038	0.0004	0.0019	0.0019	0.0019
−0.2	−0.6	0.0031	0.0031	-0.0020	0.0021	0.0021	0.0020
−0.2	−0.4	0.0016	-0.0031	-0.0018	0.0024	0.0025	0.0020
−0.2	−0.2	-0.0020	-0.0093	0.0003	0.0024	0.0023	0.0021
−0.2	0	0.0028	0.0011	-0.0014	0.0020	0.0020	0.0022
−0.2	0.2	0.0215	0.0202	-0.0011	0.0027	0.0026	0.0022
−0.2	0.4	0.0164	0.0172	-0.0018	0.0025	0.0026	0.0024
−0.2	0.6	0.0217	0.0217	0.0023	0.0026	0.0026	0.0023
−0.2	0.8	0.0180	0.0180	-0.0011	0.0025	0.0025	0.0022
0	−0.8	-0.0016	-0.0016	-0.0028	0.0021	0.0021	0.0021
0	−0.6	-0.0009	-0.0016	-0.0018	0.0021	0.0021	0.0021
0	−0.4	0.0005	-0.0044	0.0004	0.0021	0.0022	0.0020
0	−0.2	-0.0043	-0.0100	0.0001	0.0021	0.0021	0.0021
0	0	-0.0001	0.0000	-0.0016	0.0020	0.0020	0.0021
0	0.2	0.0052	0.0151	-0.0002	0.0023	0.0026	0.0023
0	0.4	-0.0001	0.0005	-0.0012	0.0021	0.0021	0.0022
0	0.6	-0.0006	-0.0006	0.0000	0.0024	0.0024	0.0026
0	0.8	0.0013	0.0013	0.0016	0.0022	0.0022	0.0023
0.2	−0.8	-0.0041	-0.0041	-0.0024	0.0022	0.0022	0.0022
0.2	−0.6	-0.0058	-0.0063	-0.0014	0.0022	0.0022	0.0021
0.2	−0.4	-0.0105	-0.0123	-0.0019	0.0022	0.0023	0.0021
0.2	−0.2	-0.0116	-0.0101	0.0003	0.0022	0.0022	0.0021
0.2	0	-0.0044	-0.0010	-0.0036	0.0021	0.0021	0.0022
0.2	0.2	-0.0048	0.0011	-0.0011	0.0023	0.0024	0.0022
0.2	0.4	-0.0166	-0.0166	-0.0045	0.0025	0.0025	0.0022
0.2	0.6	-0.0194	-0.0194	0.0007	0.0026	0.0026	0.0024
0.2	0.8	-0.0215	-0.0215	0.0006	0.0026	0.0026	0.0022
0.4	−0.8	-0.0028	-0.0028	0.0030	0.0024	0.0024	0.0023
0.4	−0.6	-0.0116	-0.0111	0.0007	0.0024	0.0024	0.0022
0.4	−0.4	-0.0154	-0.0099	0.0009	0.0026	0.0025	0.0024
0.4	−0.2	-0.0027	-0.0019	-0.0008	0.0024	0.0023	0.0023
0.4	0	-0.0018	0.0006	-0.0023	0.0021	0.0021	0.0021
0.4	0.2	-0.0091	-0.0089	-0.0021	0.0024	0.0024	0.0023
0.4	0.4	-0.0239	-0.0239	-0.0026	0.0030	0.0030	0.0024
0.4	0.6	-0.0386	-0.0386	-0.0023	0.0038	0.0038	0.0025
0.4	0.8	-0.0475	-0.0475	-0.0011	0.0044	0.0044	0.0023
0.6	−0.8	-0.0084	-0.0055	-0.0009	0.0030	0.0030	0.0025
0.6	−0.6	0.0019	0.0066	-0.0013	0.0030	0.0029	0.0025
0.6	−0.4	0.0118	0.0122	0.0009	0.0027	0.0027	0.0024
0.6	−0.2	0.0070	0.0071	-0.0022	0.0024	0.0024	0.0024
0.6	0	0.0001	0.0001	-0.0013	0.0024	0.0024	0.0024
0.6	0.2	-0.0113	-0.0113	0.0001	0.0023	0.0023	0.0022
0.6	0.4	-0.0258	-0.0258	0.0010	0.0027	0.0027	0.0022
0.6	0.6	-0.0461	-0.0461	-0.0016	0.0042	0.0042	0.0022
0.6	0.8	-0.0667	-0.0667	0.0001	0.0066	0.0066	0.0022
0.8	−0.8	0.0601	0.0603	0.0018	0.0066	0.0066	0.0024
0.8	−0.6	0.0485	0.0485	-0.0001	0.0052	0.0052	0.0024
0.8	−0.4	0.0354	0.0354	0.0007	0.0037	0.0037	0.0023
0.8	−0.2	0.0190	0.0190	0.0027	0.0027	0.0027	0.0024
0.8	0	0.0000	0.0000	0.0010	0.0022	0.0022	0.0024
0.8	0.2	-0.0179	-0.0179	0.0001	0.0025	0.0025	0.0024
0.8	0.4	-0.0351	-0.0351	-0.0003	0.0033	0.0033	0.0024
0.8	0.6	-0.0562	-0.0562	-0.0020	0.0055	0.0055	0.0024
0.8	0.8	-0.0822	-0.0822	-0.0020	0.0094	0.0094	0.0021
average		0.0215	0.0217	0.0015	0.0035	0.0035	0.0022

Table 5: Bias and MSE of $\widehat{\beta}_{2,PT1}$, $\widehat{\beta}_{2,PT2}$, and $\widehat{\beta}_{2,ML}$.

		BIAS			MSE		
ρ	λ	$\widehat{\beta}_{2,PT1}$	$\widehat{\beta}_{2,PT2}$	$\widehat{\beta}_{2,ML}$	$\widehat{\beta}_{2,PT1}$	$\widehat{\beta}_{2,PT2}$	$\widehat{\beta}_{2,ML}$
−0.8	−0.8	-0.0222	-0.0222	-0.0007	0.0027	0.0027	0.0016
−0.8	−0.6	-0.0231	-0.0231	-0.0001	0.0036	0.0036	0.0020
−0.8	−0.4	-0.0309	-0.0305	-0.0004	0.0039	0.0039	0.0021
−0.8	−0.2	-0.0255	-0.0250	0.0012	0.0024	0.0024	0.0020
−0.8	0	-0.0004	0.0001	-0.0004	0.0017	0.0017	0.0019
−0.8	0.2	0.0309	0.0309	-0.0008	0.0030	0.0030	0.0019
−0.8	0.4	0.0434	0.0468	-0.0001	0.0043	0.0046	0.0019
−0.8	0.6	0.0311	0.0311	0.0033	0.0035	0.0035	0.0020
−0.8	0.8	0.0292	0.0292	0.0035	0.0034	0.0034	0.0020
−0.6	−0.8	-0.0169	-0.0167	-0.0028	0.0024	0.0024	0.0019
−0.6	−0.6	-0.0182	-0.0180	-0.0048	0.0026	0.0026	0.0018
−0.6	−0.4	-0.0186	-0.0163	-0.0007	0.0028	0.0027	0.0018
−0.6	−0.2	-0.0162	-0.0123	0.0017	0.0024	0.0024	0.0021
−0.6	0	0.0033	0.0046	0.0030	0.0018	0.0019	0.0020
−0.6	0.2	0.0214	0.0217	-0.0015	0.0024	0.0024	0.0019
−0.6	0.4	0.0207	0.0210	0.0001	0.0027	0.0028	0.0020
−0.6	0.6	0.0197	0.0197	-0.0001	0.0026	0.0026	0.0020
−0.6	0.8	0.0190	0.0190	-0.0004	0.0027	0.0027	0.0020
−0.4	−0.8	-0.0101	-0.0101	-0.0011	0.0021	0.0021	0.0019
−0.4	−0.6	-0.0102	-0.0095	-0.0033	0.0021	0.0021	0.0018
−0.4	−0.4	-0.0107	-0.0078	-0.0022	0.0024	0.0023	0.0020
−0.4	−0.2	-0.0106	-0.0056	-0.0022	0.0021	0.0021	0.0019
−0.4	0	0.0015	0.0029	-0.0002	0.0020	0.0021	0.0021
−0.4	0.2	0.0112	0.0132	-0.0006	0.0023	0.0023	0.0021
−0.4	0.4	0.0118	0.0116	-0.0001	0.0024	0.0024	0.0022
−0.4	0.6	0.0158	0.0158	0.0020	0.0025	0.0025	0.0021
−0.4	0.8	0.0105	0.0105	-0.0025	0.0025	0.0025	0.0022
−0.2	−0.8	-0.0053	-0.0053	-0.0010	0.0021	0.0021	0.0021
−0.2	−0.6	-0.0040	-0.0036	-0.0009	0.0023	0.0023	0.0023
−0.2	−0.4	-0.0019	-0.0003	0.0007	0.0020	0.0020	0.0019
−0.2	−0.2	-0.0055	-0.0034	-0.0030	0.0021	0.0021	0.0021
−0.2	0	-0.0005	-0.0001	-0.0023	0.0021	0.0021	0.0021
−0.2	0.2	0.0076	0.0073	0.0007	0.0021	0.0022	0.0020
−0.2	0.4	0.0055	0.0057	-0.0009	0.0020	0.0020	0.0020
−0.2	0.6	0.0097	0.0097	0.0024	0.0023	0.0023	0.0021
−0.2	0.8	0.0101	0.0101	0.0030	0.0022	0.0022	0.0021
0	−0.8	-0.0009	-0.0009	-0.0010	0.0021	0.0021	0.0021
0	−0.6	0.0011	0.0012	0.0004	0.0024	0.0024	0.0024
0	−0.4	-0.0006	-0.0004	-0.0011	0.0023	0.0023	0.0023
0	−0.2	-0.0025	-0.0021	-0.0022	0.0021	0.0021	0.0021
0	0	-0.0006	-0.0006	-0.0015	0.0020	0.0020	0.0021
0	0.2	-0.0002	0.0026	-0.0015	0.0021	0.0022	0.0021
0	0.4	0.0030	0.0032	0.0025	0.0021	0.0021	0.0021
0	0.6	0.0008	0.0008	0.0009	0.0022	0.0022	0.0022
0	0.8	0.0021	0.0021	0.0021	0.0021	0.0021	0.0021
0.2	−0.8	0.0051	0.0051	-0.0005	0.0021	0.0021	0.0022
0.2	−0.6	0.0016	0.0016	-0.0023	0.0021	0.0021	0.0021
0.2	−0.4	-0.0029	-0.0031	-0.0052	0.0022	0.0022	0.0022
0.2	−0.2	-0.0004	0.0001	-0.0006	0.0021	0.0021	0.0021
0.2	0	-0.0005	0.0009	-0.0015	0.0021	0.0022	0.0021
0.2	0.2	-0.0034	-0.0012	-0.0007	0.0023	0.0023	0.0023
0.2	0.4	-0.0085	-0.0085	-0.0020	0.0021	0.0021	0.0020
0.2	0.6	-0.0114	-0.0114	-0.0030	0.0022	0.0022	0.0021
0.2	0.8	-0.0090	-0.0090	-0.0013	0.0023	0.0023	0.0022
0.4	−0.8	0.0127	0.0127	0.0003	0.0025	0.0025	0.0022
0.4	−0.6	0.0086	0.0083	0.0007	0.0022	0.0022	0.0021
0.4	−0.4	0.0030	0.0049	-0.0038	0.0022	0.0022	0.0022
0.4	−0.2	0.0055	0.0060	-0.0014	0.0025	0.0025	0.0024
0.4	0	-0.0004	0.0006	-0.0015	0.0021	0.0021	0.0021
0.4	0.2	-0.0094	-0.0093	-0.0017	0.0023	0.0023	0.0023
0.4	0.4	-0.0161	-0.0161	-0.0003	0.0024	0.0024	0.0023
0.4	0.6	-0.0189	-0.0189	0.0007	0.0025	0.0025	0.0022
0.4	0.8	-0.0198	-0.0198	-0.0018	0.0024	0.0024	0.0020
0.6	−0.8	0.0258	0.0261	0.0008	0.0033	0.0034	0.0022
0.6	−0.6	0.0274	0.0300	-0.0002	0.0032	0.0033	0.0022
0.6	−0.4	0.0227	0.0228	-0.0001	0.0028	0.0028	0.0023
0.6	−0.2	0.0146	0.0146	0.0015	0.0024	0.0024	0.0023
0.6	0	0.0007	0.0007	-0.0003	0.0022	0.0022	0.0023
0.6	0.2	-0.0138	-0.0138	-0.0012	0.0022	0.0022	0.0021
0.6	0.4	-0.0272	-0.0272	-0.0027	0.0027	0.0027	0.0021
0.6	0.6	-0.0332	-0.0332	-0.0005	0.0031	0.0031	0.0021
0.6	0.8	-0.0309	-0.0309	-0.0004	0.0030	0.0030	0.0019
0.8	−0.8	0.0780	0.0781	0.0043	0.0091	0.0091	0.0025
0.8	−0.6	0.0572	0.0572	0.0020	0.0058	0.0058	0.0022
0.8	−0.4	0.0362	0.0362	-0.0007	0.0036	0.0036	0.0022
0.8	−0.2	0.0173	0.0173	0.0006	0.0024	0.0024	0.0022
0.8	0	0.0028	0.0028	0.0041	0.0021	0.0021	0.0023
0.8	0.2	-0.0154	-0.0154	0.0015	0.0021	0.0021	0.0021
0.8	0.4	-0.0327	-0.0327	-0.0015	0.0029	0.0029	0.0021
0.8	0.6	-0.0414	-0.0414	-0.0008	0.0035	0.0035	0.0020
0.8	0.8	-0.0445	-0.0445	-0.0038	0.0042	0.0042	0.0021
average		0.0149	0.0148	0.0015	0.0026	0.0026	0.0021

Table 6: Size of Tests Associated with Pre-test and ML Estimators of the Hypothesis that a Respective Parameter is Equal to the True Parameter.

ρ	λ	$\widehat{\lambda}_{PT1}$	$\widehat{\rho}_{PT1}$	$\widehat{\beta}_{1,PT1}$	$\widehat{\beta}_{2,PT1}$	$\widehat{\lambda}_{PT2}$	$\widehat{\rho}_{PT2}$	$\widehat{\beta}_{1,PT2}$	$\widehat{\beta}_{2,PT2}$	$\widehat{\lambda}_{ML}$	$\widehat{\lambda}_{ML}$	$\widehat{\beta}_{1,ML}$	$\widehat{\beta}_{2,ML}$
−0.8	−0.8	1.0000	1.0000	0.0920	0.1050	1.0000	1.0000	0.0920	0.1050	0.0740	0.0710	0.0650	0.0320
−0.8	−0.6	1.0000	1.0000	0.2340	0.1710	1.0000	1.0000	0.2340	0.1710	0.0540	0.0620	0.0520	0.0610
−0.8	−0.4	1.0000	0.9950	0.4980	0.2310	1.0000	0.9980	0.4910	0.2290	0.0570	0.0650	0.0460	0.0580
−0.8	−0.2	1.0000	0.6930	0.3410	0.1140	1.0000	0.7050	0.3340	0.1140	0.0480	0.0550	0.0490	0.0570
−0.8	0	0.0080	0.0660	0.0570	0.0490	0.0030	0.0800	0.0560	0.0480	0.0510	0.0610	0.0500	0.0490
−0.8	0.2	1.0000	0.6820	0.3610	0.1000	1.0000	0.6820	0.3610	0.1000	0.0380	0.0440	0.0450	0.0470
−0.8	0.4	1.0000	0.9990	0.6130	0.1530	1.0000	0.9990	0.6560	0.1700	0.0470	0.0710	0.0560	0.0510
−0.8	0.6	1.0000	1.0000	0.3380	0.0870	1.0000	1.0000	0.3380	0.0870	0.0620	0.0500	0.0400	0.0500
−0.8	0.8	0.9950	1.0000	0.3060	0.0890	0.9950	1.0000	0.3060	0.0890	0.0510	0.0540	0.0580	0.0590
−0.6	−0.8	1.0000	1.0000	0.0680	0.0620	1.0000	1.0000	0.0680	0.0610	0.0720	0.0790	0.0610	0.0490
−0.6	−0.6	1.0000	1.0000	0.1410	0.1000	1.0000	1.0000	0.1420	0.0980	0.0550	0.0480	0.0460	0.0360
−0.6	−0.4	1.0000	0.9980	0.3140	0.1150	1.0000	1.0000	0.2850	0.1020	0.0580	0.0560	0.0520	0.0460
−0.6	−0.2	1.0000	0.7410	0.2460	0.0910	1.0000	0.8270	0.2090	0.0820	0.0620	0.0490	0.0430	0.0560
−0.6	0	0.0300	0.0890	0.0790	0.0460	0.0040	0.1890	0.0700	0.0430	0.0600	0.0660	0.0590	0.0470
−0.6	0.2	1.0000	0.7070	0.2410	0.0670	1.0000	0.7110	0.2410	0.0670	0.0510	0.0630	0.0440	0.0530
−0.6	0.4	1.0000	1.0000	0.2720	0.0610	1.0000	1.0000	0.2690	0.0640	0.0420	0.0510	0.0560	0.0480
−0.6	0.6	0.9920	1.0000	0.2140	0.0610	0.9920	1.0000	0.2140	0.0610	0.0490	0.0540	0.0500	0.0500
−0.6	0.8	0.9450	1.0000	0.2280	0.0700	0.9450	1.0000	0.2280	0.0700	0.0460	0.0460	0.0490	0.0550
−0.4	−0.8	0.9490	1.0000	0.0500	0.0480	0.9490	1.0000	0.0500	0.0480	0.0480	0.0510	0.0530	0.0460
−0.4	−0.6	0.9530	1.0000	0.0850	0.0540	0.9590	1.0000	0.0800	0.0480	0.0630	0.0580	0.0520	0.0310
−0.4	−0.4	0.9780	0.9990	0.1510	0.0770	0.9900	1.0000	0.1120	0.0650	0.0580	0.0620	0.0450	0.0490
−0.4	−0.2	0.9870	0.8210	0.1400	0.0610	0.9990	0.9520	0.0950	0.0500	0.0580	0.0510	0.0450	0.0460
−0.4	0	0.0810	0.1460	0.0610	0.0490	0.0040	0.5360	0.0520	0.0500	0.0540	0.0690	0.0490	0.0510
−0.4	0.2	0.9940	0.7450	0.1310	0.0590	0.9300	0.7670	0.1550	0.0600	0.0520	0.0520	0.0580	0.0550
−0.4	0.4	0.8960	1.0000	0.1220	0.0610	0.8960	1.0000	0.1170	0.0610	0.0500	0.0690	0.0500	0.0570
−0.4	0.6	0.8460	1.0000	0.1160	0.0720	0.8460	1.0000	0.1160	0.0720	0.0470	0.0370	0.0510	0.0510
−0.4	0.8	0.6410	1.0000	0.1270	0.0670	0.6410	1.0000	0.1270	0.0670	0.0540	0.0550	0.0560	0.0670
−0.2	−0.8	0.4460	1.0000	0.0440	0.0490	0.4460	1.0000	0.0440	0.0490	0.0590	0.0730	0.0470	0.0520
−0.2	−0.6	0.4750	1.0000	0.0620	0.0680	0.4890	1.0000	0.0560	0.0650	0.0590	0.0620	0.0570	0.0710
−0.2	−0.4	0.5340	1.0000	0.0840	0.0480	0.6850	1.0000	0.0800	0.0430	0.0680	0.0620	0.0550	0.0460
−0.2	−0.2	0.7250	0.8860	0.0800	0.0600	0.9480	0.9960	0.0610	0.0490	0.0540	0.0630	0.0560	0.0600
−0.2	0	0.0940	0.6320	0.0430	0.0590	0.0060	0.8890	0.0400	0.0570	0.0470	0.0580	0.0390	0.0580
−0.2	0.2	0.7820	0.9980	0.0920	0.0560	0.6760	0.9600	0.0910	0.0570	0.0490	0.0470	0.0510	0.0520
−0.2	0.4	0.3870	1.0000	0.0680	0.0460	0.3920	1.0000	0.0710	0.0460	0.0510	0.0510	0.0580	0.0500
−0.2	0.6	0.3340	1.0000	0.0690	0.0640	0.3340	1.0000	0.0690	0.0640	0.0430	0.0500	0.0550	0.0540
−0.2	0.8	0.2360	1.0000	0.0670	0.0520	0.2360	1.0000	0.0670	0.0520	0.0500	0.0490	0.0500	0.0450
0	−0.8	0.0630	0.0000	0.0650	0.0560	0.0640	0.0000	0.0650	0.0560	0.0560	0.0590	0.0650	0.0540
0	−0.6	0.0490	0.0030	0.0530	0.0650	0.0710	0.0000	0.0540	0.0630	0.0720	0.0580	0.0560	0.0670
0	−0.4	0.0760	0.0270	0.0570	0.0610	0.2830	0.0010	0.0600	0.0590	0.0500	0.0580	0.0510	0.0630
0	−0.2	0.4740	0.1230	0.0650	0.0520	0.7960	0.0040	0.0590	0.0510	0.0620	0.0610	0.0500	0.0540
0	0	0.0260	0.0170	0.0440	0.0390	0.0080	0.0070	0.0450	0.0400	0.0580	0.0570	0.0460	0.0410
0	0.2	0.2990	0.1480	0.0580	0.0450	0.6220	0.0180	0.0820	0.0460	0.0640	0.0560	0.0520	0.0420
0	0.4	0.0840	0.0250	0.0470	0.0500	0.0930	0.0210	0.0480	0.0520	0.0500	0.0610	0.0540	0.0530
0	0.6	0.0600	0.0050	0.0620	0.0440	0.0600	0.0050	0.0620	0.0440	0.0520	0.0430	0.0650	0.0510
0	0.8	0.0560	0.0000	0.0530	0.0470	0.0560	0.0000	0.0530	0.0470	0.0490	0.0560	0.0490	0.0470
0.2	−0.8	0.5280	1.0000	0.0610	0.0650	0.5280	1.0000	0.0610	0.0650	0.0590	0.0590	0.0540	0.0600
0.2	−0.6	0.5240	1.0000	0.0570	0.0390	0.5260	1.0000	0.0600	0.0390	0.0570	0.0710	0.0440	0.0380
0.2	−0.4	0.5220	1.0000	0.0660	0.0590	0.5710	0.9980	0.0680	0.0570	0.0620	0.0730	0.0520	0.0540
0.2	−0.2	0.9540	0.9670	0.0530	0.0460	0.8420	0.8730	0.0530	0.0460	0.0670	0.0700	0.0500	0.0500
0.2	0	0.1420	0.4880	0.0430	0.0470	0.0250	0.7560	0.0420	0.0470	0.0640	0.0700	0.0460	0.0450
0.2	0.2	0.8040	0.9140	0.0670	0.0590	0.8970	0.9690	0.0710	0.0620	0.0690	0.0720	0.0600	0.0550
0.2	0.4	0.6740	1.0000	0.0670	0.0490	0.6740	1.0000	0.0670	0.0490	0.0640	0.0600	0.0580	0.0490
0.2	0.6	0.5630	1.0000	0.0760	0.0600	0.5630	1.0000	0.0760	0.0600	0.0610	0.0660	0.0590	0.0500
0.2	0.8	0.4290	1.0000	0.0660	0.0650	0.4290	1.0000	0.0660	0.0650	0.0360	0.0430	0.0470	0.0540
0.4	−0.8	0.9890	1.0000	0.0600	0.0770	0.9890	1.0000	0.0600	0.0770	0.0540	0.0430	0.0490	0.0490
0.4	−0.6	0.9980	1.0000	0.0620	0.0520	0.9980	1.0000	0.0600	0.0520	0.0700	0.0600	0.0430	0.0400
0.4	−0.4	0.9910	0.9830	0.0740	0.0470	0.9910	0.9830	0.0630	0.0480	0.0780	0.0710	0.0530	0.0510
0.4	−0.2	1.0000	0.5930	0.0650	0.0630	0.9980	0.6090	0.0660	0.0630	0.0650	0.0690	0.0540	0.0580
0.4	0	0.0850	0.1370	0.0440	0.0500	0.0440	0.2100	0.0490	0.0530	0.0790	0.0780	0.0450	0.0500
0.4	0.2	0.9990	0.8380	0.0490	0.0670	1.0000	0.8400	0.0500	0.0670	0.0870	0.0770	0.0430	0.0590
0.4	0.4	0.9970	1.0000	0.0940	0.0770	0.9970	1.0000	0.0940	0.0770	0.0660	0.0640	0.0510	0.0570
0.4	0.6	0.9930	1.0000	0.1470	0.0720	0.9930	1.0000	0.1470	0.0720	0.0650	0.0640	0.0560	0.0590
0.4	0.8	0.9580	1.0000	0.1770	0.0580	0.9580	1.0000	0.1770	0.0580	0.0470	0.0530	0.0490	0.0390
0.6	−0.8	1.0000	1.0000	0.0840	0.0900	1.0000	1.0000	0.0840	0.0890	0.0390	0.0510	0.0610	0.0510
0.6	−0.6	1.0000	1.0000	0.0770	0.0840	1.0000	1.0000	0.0740	0.0860	0.0700	0.0720	0.0660	0.0420
0.6	−0.4	1.0000	0.9830	0.0600	0.0790	1.0000	0.9830	0.0600	0.0790	0.0650	0.0580	0.0460	0.0530
0.6	−0.2	1.0000	0.5310	0.0620	0.0580	1.0000	0.5310	0.0630	0.0580	0.0690	0.0600	0.0510	0.0440
0.6	0	0.0100	0.0740	0.0640	0.0510	0.0100	0.0740	0.0640	0.0510	0.0860	0.0810	0.0600	0.0600
0.6	0.2	1.0000	0.7510	0.0580	0.0590	1.0000	0.7510	0.0580	0.0590	0.0900	0.0810	0.0460	0.0450
0.6	0.4	1.0000	1.0000	0.0810	0.0960	1.0000	1.0000	0.0810	0.0960	0.1020	0.1080	0.0540	0.0500
0.6	0.6	1.0000	1.0000	0.1670	0.1240	1.0000	1.0000	0.1670	0.1240	0.1000	0.0940	0.0500	0.0510
0.6	0.8	1.0000	1.0000	0.2940	0.0930	1.0000	1.0000	0.2940	0.0930	0.0980	0.0910	0.0420	0.0370
0.8	−0.8	1.0000	1.0000	0.2160	0.3260	1.0000	1.0000	0.2170	0.3270	0.0400	0.0540	0.0340	0.0620
0.8	−0.6	1.0000	1.0000	0.1620	0.2150	1.0000	1.0000	0.1620	0.2150	0.0500	0.0490	0.0510	0.0570
0.8	−0.4	1.0000	0.9590	0.1110	0.1150	1.0000	0.9590	0.1110	0.1150	0.0430	0.0430	0.0400	0.0570
0.8	−0.2	1.0000	0.4970	0.0640	0.0560	1.0000	0.4970	0.0640	0.0560	0.0640	0.0600	0.0400	0.0430
0.8	0	0.0050	0.0510	0.0510	0.0540	0.0050	0.0510	0.0510	0.0540	0.0650	0.0640	0.0490	0.0570
0.8	0.2	1.0000	0.6840	0.0680	0.0550	1.0000	0.6840	0.0680	0.0550	0.0800	0.0850	0.0500	0.0450
0.8	0.4	1.0000	0.9990	0.1080	0.1160	1.0000	0.9990	0.1080	0.1160	0.1290	0.1170	0.0510	0.0490
0.8	0.6	1.0000	1.0000	0.2200	0.1320	1.0000	1.0000	0.2200	0.1320	0.1400	0.1150	0.0550	0.0390
0.8	0.8	1.0000	1.0000	0.3570	0.1510	1.0000	1.0000	0.3570	0.1510	0.1650	0.1710	0.0360	0.0470
average		0.7119	0.7654	0.1280	0.0776	0.7217	0.7792	0.1264	0.0770	0.0630	0.0641	0.0510	0.0509

Appendix: Mont Carlo Estimates of Size of Tests

In the following we provide more details on computation of the size of the tests under the null hypothesis, where the null hypothesis is that the parameters are equal to their true values. We discuss this exemplarily for testing the null hypothesis that $\lambda = \lambda_0$.

In the following let $\widehat{\sigma}_{\widehat{\lambda}_{MLE}}$, $\widehat{\sigma}_{\widehat{\lambda}_{MLL}}$ and $\widehat{\sigma}_{\widehat{\lambda}_{ML}}$denote the estimate for the standard deviation of the ML estimator calculated from the negative inverse Hessian of the log-likelihood function of the spatial error model (3), the lag model (5) and the encompassing model (7), respectively. Also, let m be the index for the Monte Carlo experiment, and let M be the total number of Monte Carlo experiments.

First consider the pre-test estimators in the case where $\lambda_0 \neq 0$. In this case the researcher would typically reject $H_0 : \lambda = \lambda_0$ if the pre-test procedure selects the spatial lag model and the t-test statistic exceeds the .95 fractile of its asymptotic standardized normal distribution. However, implicitly the researcher also rejects $H_0 : \lambda = \lambda_0$ if the pre-test procedure selects the OLS model, or the spatial error model. Thus we estimate the size of the test associated with the pre-test procedure as

$$\frac{1}{M}\sum_{m=1}^{M}\left\{\mathbf{1}\left[\widehat{\lambda}_{PT}^{(m)} = \widehat{\lambda}_{MLL}^{(m)}\right]\mathbf{1}\left[\left|\frac{\widehat{\lambda}_{PT}^{(m)} - \lambda_0}{\widehat{\sigma}_{\widehat{\lambda}_{MLL}}^{(m)}}\right| > 1.96\right] + \mathbf{1}\left[\widehat{\lambda}_{PT}^{(m)} = \widehat{\lambda}_{OLS}^{(m)}\right] + \mathbf{1}\left[\widehat{\lambda}_{PT}^{(m)} = \widehat{\lambda}_{MLE}^{(m)}\right]\right\}.$$

Next consider the case $\lambda_0 = 0$. Since $H_0 : \lambda = 0$ is compatible with the OLS and spatial error model, we would reject $H_0 : \lambda = 0$ only if the pre-test procedure selects the spatial lag model and the t-test statistic exceeds the .95 fractile of the standardized normal. Thus the size of the test can be estimated as

$$\frac{1}{M}\sum_{m=1}^{M}\left\{\mathbf{1}\left[\widehat{\lambda}_{PT}^{(m)} = \widehat{\lambda}_{MLL}^{(m)}\right]\mathbf{1}\left[\left|\frac{\widehat{\lambda}_{PT}^{(m)} - \lambda_0}{\widehat{\sigma}_{\widehat{\lambda}_{MLL}}^{(m)}}\right| > 1.96\right]\right\}.$$

Next consider the ML estimator of the encompassing model. In this case the size of the test is estimated as usual by

$$\frac{1}{M}\sum_{m=1}^{M}\mathbf{1}\left[\left|\frac{\widehat{\lambda}_{ML}^{(m)} - \lambda_0}{\widehat{\sigma}_{\widehat{\lambda}_{ML}}^{(m)}}\right| > 1.96\right].$$

For testing $H_0 : \rho = \rho_0$ the size of the test is computed in an analogous fashion.