Abstract

A model of two-product, Hotelling duopolists is examined in which firms can engage in mixed bundling. Even though products are independent in consumption, bundled pricing induces complementarity across different products. The most efficient outcome, symmetric independent goods pricing, is no longer an equilibrium when mixed bundling is feasible. On the other hand, inefficient pure bundling equilibria are robust to mixed bundling pricing because a pricing externality may prevent firms from independently achieving more attractive variety of consumer constructed bundles. Mixed bundle pricing equilibria typically exist as well. While these outcomes yield higher surplus than the pure bundling equilibria, they are still inefficient. Thus, if mergers of firms increase the likelihood of mixed bundle pricing, these mergers can be harmful even if the firms do not initially compete. Blended market structures where two single product firms compete against a two-product firm are also examined and equilibria exist where a merger of the two single product firms can enhance both consumer welfare and total welfare.

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Mixed Bundling and Imperfect Competition

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1 Introduction

There are very few significant single-product firms.[1] Nevertheless, most of what we know about equilibria among imperfectly competitive firms assumes this property. This assumption would be harmless, if it were true that, at least when the firms' different products compete in different markets, equilibrium behavior could be analyzed separately, that is, market by market. However, once firms recognize that they can set prices, not only for individual products but for combinations of products also, markets that may be independent on the basis of intrinsic preferences, become related through market pricing. This pricing linkage has consequences for welfare, for anti-trust policy, and for equilibrium predictions of firm behavior.

Multi-product firms frequently choose a rich variety of pricing policies. Telecommunications firms offer a la carte pricing as well as bundled pricing for broadband, cell-phone and video services. Insurance firms offer separate pricing for stand-alone policies, as well as discounts for bundling house, car and life insurance policies. Stores with frequent buyer programs offer effective bundled discounts when buyers earn credits based on total dollar purchases. It has long been recognized that bundled pricing enables firms with some power over price to induce privately informed buyers to separate into endogenously formed market segments (and thus is a form of what used to be termed, 'second-degree price discrimination'.[2]) Recent theoretical work has furthered our understanding of such pricing incentives for monopolies (see, for example, Manelli and Vincent 2006 and 2007) – very little work has focused on mixed bundled pricing among imperfectly competitive firms.

In this paper, I examine a simple Hotelling type of competition between two, two-product firms (Firms A and B both sell products 1 and 2) and illustrate the incentives for, and consequences of, mixed bundled pricing. Equilibrium behavior when bundled pricing is not feasible, reproduces the standard Hotelling outcome and results in total surplus maximizing partitions of the two markets. When only pure bundle prices are feasible (the two different products of each firm can only be sold as a pair), there is a socially inefficient loss of variety but despite this inefficiency, many consumers gain because of heightened competition between firms. When firms market their

[1]While single product firms make up about 59% of all U.S. firms, multi-product firms account for 91% of output and firms present in multiple sectors account for 76% of output. See Bernard, et. al.

[2]Stole (2007) suggests the more informative term 'endogenous price discrimination'. He also provides a thorough survey of the current literature on price discrimination in competitive markets.

products both as bundles and as independent goods, the efficient independent goods pricing equilibrium is no longer an equilibrium. However, the equilibrium outcome corresponding to pure bundled pricing very frequently remains an equilibrium despite the possibility of selling goods independently. The reason for this is that, even though goods may be neither substitutes nor complements intrinsically, once one firm sells its two products as a bundle, product 1 of, say, Firm A becomes a 'competitive complement' with product 2 of Firm B so that a buyer considering buying product 1 alone from A can also get 2 from B and construct a viable alternative to either firm's bundle. If Firm B does not offer product 2 independently. Firm A may also choose not to offer product 1 independently.

As well as the pure bundling equilibrium, there is also often a mixed bundling equilibrium, where both firms sell products as bundles and as independent goods. I provide necessary conditions for a symmetric mixed bundling equilibrium and provide examples where such conditions are also sufficient. The equilibrium outcome corresponding to non-trivial mixed bundling provides more product variety than the pure bundling equilibrium and is, in that sense, more efficient. Equilibrium prices tend to be higher than pure bundle pricing and, in that sense, is both more attractive to firms and less attractive to consumers. The equilibria of these pricing games have, to my knowledge, been studied very little, even though, casual observation indicates that many firms utilize such strategies.[3] The technical hurdles to solving such equilibria are significant and stem, in part, from the same reasons multiple equilibria arise – products that under independent goods pricing are strategically independent, when competing against bundles, may become strategic substitutes while goods that are strategic complements originally, become strategically independent. As a consequence, games where independent goods pricing or pure bundled pricing are supermodular, are not supermodular games when mixed bundled pricing is possible.

Since bundled pricing is a form of price discrimination, this paper can also be understood as a model of price discrimination in a competitive environment. In other models of competitive price discrimination, such as Armstrong and Vickers (2001), consumers become customers of only one firm which may compete for its clients via discriminating price menus. In these environments, as Armstrong and Vickers note, firms effectively compete in net utilities – a single-dimensional variable. Here, in

[3]Chen (1997) provides an early study of mixed bundling under competition, however, in his model, bundling is motivated by a desire to lessen competition by product differentiation. A consequence is that non-trivial mixed bundling tends to be less profitable for his firms as the ability to differentiate is lessened.

contrast, consumers are free to and often will purchase products from different firms. The frontier on which the competing firms interact is multi-dimensional. The firms compete with each other directly with their bundles but also compete with their own bundles and their rival's bundle with bundles that consumers form using each rival's independently sold products. The scope for using mixed bundling to induce separation among privately informed consumers requires an implicit cooperation with a rival who is also competing for the same consumers.

2 Model

Two firms, A, B, produce and sell two products, $j = 1, 2$ each at constant marginal cost, c_j. Consumers have inelastic demand for one unit of each good and are differentiated according to a two dimensional type, $(x_1, x_2) \in [0,1]^2$. Let $q_1^i \in \{0, 1\}$ denote the purchases of a consumer of a good of type l from Firm i. (Note that inelastic demand implies that, typically, $q_l^i q_l^j = 0$ – for most price profiles, we can expect consumers to buy no more than one unit of a product.) A consumer located at (x_1, x_2) who makes purchases $q_l^i, i = A, B, l = 1, 2$ and pays a total of m for the purchase, receives utility

$$u(q_1^A, q_2^A, q_1^B, q_2^B, m; x_1, x_2) = \sum_{j=1}^{2} (V^j - t_j x_j)\mathbf{1}_{\{q_j^A > 0\}} + (V^j - t_j(1-(x_j))\mathbf{1}_{\{q_j^B > 0\}} - m. \quad (1)$$

Consumer types are distributed uniformly and independently on $[0,1]^2$. The distribution generates a measure $\mu(F)$ on (measurable) subsets $F \subset [0,1]^2$ which is equal to the area of F. I do not impose symmetry across costs or the differentiation parameters. For concreteness, assume that good 1 is more differentiated than good 2, so that $t_1 \geq t_2 \geq 0$. Finally, it is assumed that

$$V^j \geq 2t_j + c_j. \quad (2)$$

This condition is sufficient to ensure that the equilibria analyzed in this paper have the Hotelling property that the firms compete on the margin for each other's consumers.

This model of multiple horizontally differentiated products is similar to the model introduced in Matutes and Regibeau (1988). In their analysis, which is a study of systems competition, consumers must purchase one of each product in order to obtain utility. Thus, products are, by construction, strong complements. Here, products are independent in consumption. Purchasing one product does not affect a consumer's willingness to pay for the other product.

Firms are assumed to compete by simultaneously choosing prices. Either because consumer types are privately known or because of regulation, it is assumed that prices may not be conditioned on consumer type. For any profile of prices, consumers then select which goods to buy from which firms.[4]

2.1 Strategic Modes

Multi-product firms often have the option of selling their goods priced separately (independent goods pricing), only as a bundle (pure bundling) or both (mixed bundling). The pricing game for the different cases require different strategy spaces for the firms which I will refer to as different strategic modes. When firms engage in pure bundling, they simultaneously select a real number corresponding to the price of the bundle. When firms engage in mixed bundling, they simultaneously select an ordered triple, for example, (p_1^A, p_2^A, P^A) represents the three prices a consumer pays depending on which subset she selects from Firm A. If she buys good j independently, she pays p_j^A whereas if she buys the pair together as a bundle, she pays P^A. In all three pricing games, it is assumed that all consumers observe the prices offered by both firms and may select any combination of purchases at the offered prices from any firm to maximize (1). In certain circumstances, if the firm can monitor consumer purchases, then it is conceivable that the firm could require any consumer who buys both goods to pay a total price larger than the sum of the individual goods prices. In this analysis, it is assumed that the firm cannot monitor purchases. Thus, if a firm engages in mixed bundling, if the bundle price is such that $P^i > p_1^i + p_2^i$, no sales of the bundle are made, though, consumers may purchase both goods from the same firm by paying the independent good price for each good. This feature potentially introduces a trivial multiplicity of equilibria. I consider two equilibria to be essentially equivalent if both equilibria generate the same consumer utility, sales and firm profits.

Observe that the interpretation of firms' locations is more flexible than the literal model suggests. The 'location' of Firm A's product 1 is not literally the same as the 'location' of its product 2. All that is required is that a consumer at (x_1, x_2) choosing to purchase Firm A's product j, incurs a differentiation cost relative to purchasing Firm B's product j of $t_j x_j$ versus $t_j(1 - x_j)$. Indeed, the only thing that ties Firm

[4]For a multi-product monopolist, Manelli and Vincent (2006) show that posted price schedules are dominated by more complex trading mechanisms involving randomization. However, in the case of uniformly distributed types, they show that posted price mechanisms are, in fact, optimal among all mechanisms. In this paper, I only consider posted prices.

A's product 1 to its product 2, is the fact that they may be priced together. The strategic mode of independent goods pricing is thus simply Hotelling competition in two independent markets and could represent equilibrium outcomes with four single-product firms, $(A1, A2, B1, B2)$ instead of two two-product firms. The change in strategic mode from independent goods pricing to mixed bundled pricing might reflect, then, one of the consequences of a pair of conglomerate mergers, where two pairs of horizontally unrelated firms, $A1, A2$, and $B1, B2$ merge and now are incentivised to engage in richer pricing strategies.

Given that mixed bundling in the context of this model requires only the marketing decision to price purchases of a pair of goods at the same time as pricing the goods independently, the most appropriate strategic mode for firms already selling multi-products independently is mixed bundling. Since both independent goods pricing and pure bundling are feasible actions for the firms in the mixed bundling mode, one can then investigate whether or not independent goods pricing or pure bundling arises endogenously as equilibrium behavior (in part, the topic of Section 4). There are market environments where each strategic mode may arise initially and then may change to a different mode exogenously. The description of four independent firms pairwise merging is an example of a move from the independent goods mode to mixed bundling. Whinston (1990) argues that pure bundling can arise as consequence of technological constraints. For example Microsoft argued that the 'forced' bundling of its browser and operating system arose because of each component's need to integrate fully with the other. [5] Of course, further software advances (or encouragement from anti-trust authorities) could later make it feasible for the firm to price the products independently as well. Government restrictions may also constrain firms from adopting full mixed bundling. Finally, it is possible that firms themselves only gradually learn the advantages of other strategic modes. So, while telecommunication firms may have initially operated wireless and broadband divisions separately, experience and market experiments may have revealed the greater advantages of richer pricing schemes.

[5] See Rubenfeld, (2014) p. 630.

3 Equilibrium in Independent Goods Pricing and Pure Bundle Pricing

If firms can only post individual goods prices for each product, each market is the standard single good Hotelling game with linear costs and uniformly distributed consumers. Equilibrium prices are thus [6]

$$\hat{p}_j = t_j + c_j.$$

Each market is evenly divided between the two firms and prices reflect each product's costs and degree of differentiation.

Now, suppose that because of design restrictions or for other reasons, firms can only sell their products as a bundle of two goods. Thus, if a consumer purchases from Firm A, for example, it acquires both product 1 and 2. When the two firms select the bundle prices simultaneously, then, the result is effectively, a standard Hotelling game with single products, though, now the distribution of consumer types is generated by convolution of the two random variables, $x_1 + x_2$. When consumer types are distributed on a subset of $\Re^2$, conditions that ensure that the pricing game is supermodular even if firms only sell their products as a bundle are not obvious (even the uniform distribution does not imply this property. The next theorem characterizes the equilibrium for the uniform case where the market is covered. It generalizes slightly a result shown in Matutes and Regibeau (1988).

Theorem 1. $(c_1 + c_2 + t_1, c_1 + c_2 + t_1)$ *is a symmetric equilibrium of the pure bundle pricing game*

Proof. See Appendix. □

Observe that the equilibrium bundle price depends solely on the marginal costs and the highest differentiation parameter. As long as the market is covered under both pricing modes, total surplus is maximized by minimizing differentiation costs. That objective is achieved under independent goods pricing whereas pure bundling imposes some costs from the lack of variety. However, all consumers gain under pure bundling.

Corollary 1. *All consumers have higher utility under pure bundling than independent goods pricing.*

[6] See, for example Tirole 1988, p. 280. Condition (2) ensures that firms are on a competitive margin.

Proof. Clearly, since prices are lower under pure bundling, consumers that buy the same pair under either mode are better off under pure bundling. Consider consumers who buy product 1 from Firm A and 2 from Firm B under independent goods pricing. Consumers who then switch to Firm A's bundle under pure bundling obtain a change in utility given by

$$V^1+V^2-t_1x_1-t_2x_2-t_1-c_1-c_2-(V^1+V^2-t_1x_1-t_2(1-x_2)-t_1-t_2-c_1-c_2) = 2t_2(1-x_2)$$

which is positive. The same consumers under independent goods pricing who switch to Firm B's bundle obtain a change in utility given by

$$V^1+V^2-t_1(1-x_1)-t_2(1-x_2)-t_1-c_1-c_2-(V^1+V^2-t_1x_1-t_2(1-x_2)-t_1-t_2-c_1-c_2) = t_2-t_1(1-2x_1).$$

The manifold that separates the two bundle markets satisfies $x_1 = 1/2 + t_2/(2t_1) - x_2t_2/t_1$ so these same consumers under independent goods pricing who switch to Firm B's bundle must have $x_1 \geq 1/2 - t_2/(2t_1)$ and thus the right side satisfies

$$t_2 + t_1(1 - 2x_1) \geq t_2 - t_1 + 2t_1(1/2 - t_2/(2t_1) \geq 0.$$

A similar argument holds for consumers who purchased Firm B's product 1 and Firm A's product 2. □

4 Mixed Bundle Pricing

Suppose firms are allowed to engage in mixed bundling for their goods, 1 and 2, perhaps because of a merger, a relaxation of constraints on bundled pricing or simply because firms become aware of the richer pricing options.[7] With the option of constructing either a bundle from a single firm or a bundle made up of products from both firms, the market can partition in a variety of ways. In what follows, attention is restricted to the case where $P^i \leq p_1^i + p_2^i$ so that sales of the pair of goods will

[7] ATT operated its original wireless service well before it completed its acquisition of Cingular Wireless in 2007. Its U-Verse broadband video services was rolled out throughout 2006 and 2007. Nevertheless it was not until August 2009 that it launched its first pricing plan offering cellphone and U-Verse together. A few months after this announcement, Verizon Wireless announced plans to bundle its broadband service FIOS with cellphone plans.'AT&T Plugs Wireless As Part of U-Verse TV Bundle', http://multichannel.com/news/telco-tv/att-plugs-wireless-part-u-verse-tv-bundle/297229, August, 2009. 'Introducing New Service Bundles with Wireless', http://forums.verizon.com/t5/Verizon-at-Home/Introducing-New-Service-Bundles-with-Wireless/ba-p/94395, October, 2009.

occur through the bundle price. Even if it is assumed that the market is covered, the market can segment in qualitatively different ways as the next lemma illustrates. Let AB denote the set of consumers who buy good 1 from firm A and 2 from firm B. Sets BA, AA, and BB are defined similarly. The set $A1$ is the set of consumers who buy Firm A's product 1 only. the other single product sets are defined similarly.

Lemma 1. *Suppose the menu of prices is* $\{(p_1^A, p_2^A, P^A), (p_1^B, p_2^B, P^B)\}$ *and are such that every consumer buy at least one good. Define*

$$\begin{aligned} \underline{x}_j &= \frac{t_j - (p_j^A + p_{j'}^B - P^B)}{2t_j} \\ \bar{x}_j &= \frac{t_j + p_{j'}^A + p_j^B - P^A}{2t_j}. \end{aligned}$$

Consumers separate into the intersection of $[0,1]^2$ *with the following sets:*

(i) $AB = \{(x_1, x_2) | x_2 \geq \bar{x}_2, x_1 \leq \underline{x}_1\}, BA = \{(x_1, x_2) | x_1 \geq \bar{x}_1, x_2 \leq \underline{x}_2\}.$

(ii) $AA = \{(x_1, x_2) | x_2 \leq \bar{x}_2, x_1 \leq \bar{x}_1, x_2 \leq \frac{t_1 + t_2 + P^B - P^A - 2t_1 x_1}{2t_2}\}.$

(iii) $BB = \{(x_1, x_2) | x_2 \geq \underline{x}_2, x_1 \geq \underline{x}_1, x_2 \geq \frac{t_1 + t_2 + P^B - P^A - 2t_1 x_1}{2t_2}\}.$

(iv) $A1 = \{(x_1, x_2) | x_2 \geq \frac{V^2 - (P^A - p_1^A)}{t_2}, x_2 \leq \frac{t_1 + t_2 + P^B - p_1^A - V^2 - 2t_1 x_1}{t_2}\}$

Proof. The conditions i) through iv) are derived by determining the intersection of half-spaces where each bundle choice dominates. Note that a necessary condition for the set $A1$ to have positive measure is that $p_2^B \geq V^2$. □

The manifold that separates consumers of (say) the bundle made up of Firm A's product 1 and Firm B's product two from those who buy Firm B's bundle is a vertical line. The line that separates purchasers of the firms' bundles has slope $-t_1/t_2$. However, if there exist consumers who purchase only Firm A's product 1, the manifold separating this set of consumers from those who buy Firm B's bundle has slope $-2t_1/t_2$. This property will be important in explaining why pure bundle pricing can remain an equilibrium even if mixed bundling is feasible.

Observe that if $P^i = p_1^i + p_2^i$ for both firms, i, then $\bar{x}_j = \underline{x}_j$ and the market partitions into four rectangles corresponding to only independent goods pricing. If, instead, $P^i < p_1^i + p_2^i$ for at least one firm, i, then the set of agents who buy one good from each firm is a rectangle while the set of agents who buy both goods from a single firm is a five sided figure. Suppose that $P^A \in [P^B - (t_1 - t_2), P^B + (t_1 - t_2)]$

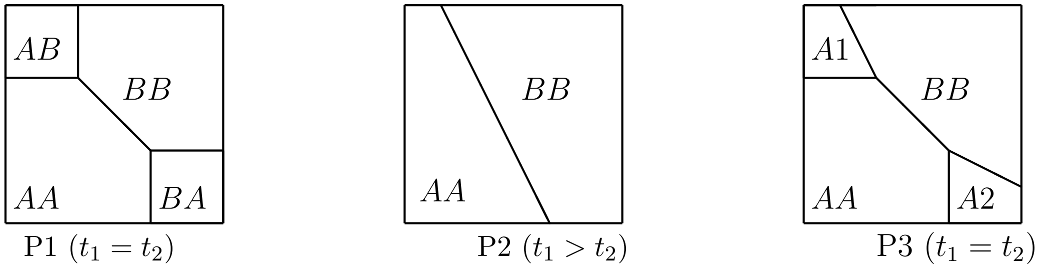

Figure 1: Examples of Market Segmentations

so the manifold separating the two bundles intersects the top and bottom edge of the support of buyer types. As $\bar{x}_2$ approaches one, then the set of agents who buy 2 from firm B and 1 from firm A vanishes. If individual good prices are such that both of these synthetic bundle sets vanish, the market converges to a pure bundling solution.

Figure 4 illustrates three main different profiles. In the standard profile, P1, all four bundle combinations are purchased $-\{AA, AB, BB, BA\}$. In profile P2, only the bundles are purchased – $\{AA, BB\}$. Finally, in profile P3, because (say) Firm B charges too high a price for its independent goods, $p_j^B > V^j$, only bundles from the same firm are purchased and only single products from, in this case, Firm A are purchased – $\{AA, A1, A2, BB\}$. Obviously, there are a variety of permutations of profile P3 that could arise.[8] Lemma 1 illustrates how the market is partitioned for a given profile of mixed bundle prices. The distribution of consumers along with this partition then generates a demand for each product for each firm corresponding to the measures of each set, $\mu(AB), \mu(AA), \mu(BA), \mu(BB)$. Therefore the profit function for each firm is easily constructed and this function can now be used to determine best responses for the firms in various mixed bundle modes. Observe that if we define the profit margin for each product as

$$\tilde{p}_j^i = p_j^i - c_j, \tilde{P}^i = P^i - c_1 - c_2,$$

the sets AB etc. can be described entirely in terms of these objects instead of prices and, so, with full market coverage and when all consumers buy both goods, the profit function can expressed solely in terms of profit margins rather than prices and costs.

If firms can engage in mixed bundle pricing, the next theorem demonstrates that the possibility of mixed bundling destroys the symmetric independent goods pricing equilibrium. The result is due to McAfee, McMillan and Whinston (1989) (MMW) who show that independent goods pricing is generally dominated by mixed goods

[8]It is also conceivable that price profiles are such that all mixed bundles are sold *and* some consumers buy only individual products. This would require (say) p_2^B close to but below V^2, P^A fairly high and p_1^A fairly low. I do not consider equilibria where this might occur.

pricing for a monopolist and argue that the same property will hold for any multi-product firm with some power over price. The idea is that, if individual pricing were to be an equilibrium, then the prices must satisfy the first order conditions characterizing each firm's best response in prices. At these prices, if Firm A, for example, were to keep the price of the two products sold as a bundle the same as the sum of the equilibrium prices and then slightly raise the individual price of product 1, the local loss in profits on product 1 sales to Firm B is of second order (by the optimality of a best response), however, the gain from shifting some consumers from buying AB to buying AA is strictly positive. For Theorem 2 only, I generalize the model to allow any symmetric, independent distribution of consumer types that generate a symmetric equilibrium of the individual goods pricing game.

Theorem 2. *(MMW) Suppose* $f_j(1/2 - x_j) = f_j(1/2 + x_j), j = 1, 2$ *and let* $(\hat{p}_1, \hat{p}_2)$ *be a symmetric Nash equilibrium of the independent goods pricing game such that the market is covered. The price profile,* $(\hat{p}_1, \hat{p}_2, \hat{p}_1 + \hat{p}_2)$ *is not an equilibrium of a game where either firm is allowed to offer mixed bundle prices.*

Theorem 2 indicates that when duopolistic firms have or acquire the ability to market their multiple competing products as a cheaper bundle, then each firm has an incentive to offer its products as a bundle. In simple Hotelling type models where the market is covered, the only determinant of efficiency is through the optimal allocation of differentiated products among consumers with different tastes. If a symmetric equilibrium exists in the independent goods pricing game, then this equilibrium achieves the social optimum given the location of firms. Thus, when one or both firms are able to engage in mixed bundling, Theorem 2 demonstrates that any equilibrium of this new game must result in a loss of efficiency relative to the independent goods pricing game. Thus if the acquisition of a firm producing A1 by a firm that produces A2 – a merger of horizontally unrelated firms – introduces this new pricing dimension an additional source of competitive harm arises.

The theorem also underlines the significance of modeling the strategic mode as either endogenous or exogenous. In Matutes and Regibeau (1992), it is assumed that firms can first commit to a particular strategic mode and then, conditional on that commitment, a pricing game ensues. In Gans and King (2006) it is assumed that cooperating firms first commit to a bundle discount and then compete on individual prices. Since individual pricing typically generates higher prices when the competition is Hotelling-like, both papers conclude that multi-product firms will often unilaterally choose the individual pricing mode. However, while this outcome seems plausible in the case of four independently optimizing firms, it is not obvious how competing multi-product firms could make such a commitment credible. Theorem 2 illustrates that in the absence of such a strong ability to commit, if multi-product firms are not restricted from mixed bundling, we should expect them to utilize it whenever it is in their (short-term) best interest.

It is possible that firms might initially find it necessary to price their products as a bundle only but then acquire the capability of selling the products also independently. For example, technological factors might have initially required selling the products together while subsequent advances enabled decomposing the bundle and selling the products separately as well. In such a case, could the original pure bundling price equilibrium survive? Recall that if the individual goods price for j exceeds V^j, no consumer will buy that product alone. If a rival firm does not offer a feasible price for one of its stand-alone products, then the other firm can only sell its stand-alone product on its own. The next lemma characterizes the prices that would have to be offered in order to make sales.

Lemma 2. *Suppose the menu of prices is* $\{(p_1^A, p_2^A, P^A), (V^1, V^2, P^B)\}$ *with* $P^i \leq p_1^i + p_2^i$*, and* $2(V^1 + V^2) - t_1 - t_2 \geq P^A + P^B$*. Consumers buy some of Firm A's product* j *if and only if*

$$p_j^A < min\{(t_1 + t_2 + P^B + P^A - 2V^2)/2, t_2 + P^A - V^2\} \tag{3}$$

Proof. See Appendix. □

The next result uses Lemma 2 to show that, in the symmetric uniform case, it remains an equilibrium for both firms to sell only the bundle.

Theorem 3. *Suppose* $t_j = t, c_j = c$*. Then* $((V^1, V^2, t + 2c), (V^1, V^2, t + 2c))$ *is an equilibrium of the mixed bundling game.*

Proof. See Appendix. □

The idea of the proof of Theorem 3 can be seen in Figure 4. When Firm A attempts to sell its product A alone, it captures consumers in set F from consumers of Firm B's bundle and it loses consumers of its own bundle in the larger set E. Lemma 2 indicates that the price that would induce any such sales is bounded above and with that bound, such an attempt is never profitable for the firm.

Bundle pricing is robust to independent goods pricing in this environment because of a coordination effect. Even though goods are consumed independently, when the

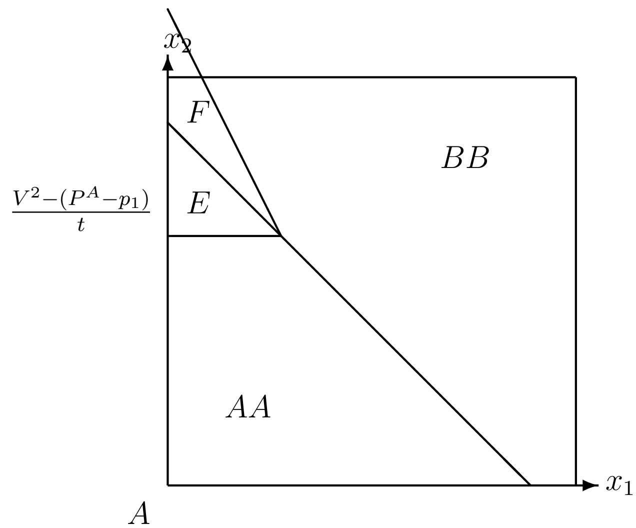

Figure 2: Proof of Theorem 3

rival firm is not offering a feasible pricing option for the other good, in order to sell its stand-alone product, a firm must offer it at a very low price in order to compete with the rival bundle. The sales generated by this strategy are not sufficient to outweigh the loss in sales from its own bundle to the stand-alone good.

Matutes and Regibeau (1992) (Lemma 1) argue that when the two products are strong complements, pure bundling is dominated by individual components pricing. One might think that, with independent goods, the coordination problem would be even less. However, in the framework here, if the rival firm is only offering its products as a bundle, the other firm does not wish to offer its own product singly because doing so cannibalizes its own bundle sales by more than it steals market share from the rival's bundle.[9] A similar cannibalization effect is present in the case of a bundling monopolist. However, unlike in the monopolist environment, in this model, the opportunity to expand market sales by independent goods pricing is mitigated because consumers in a duopoly market also have the option of purchasing the bundle from the rival firm. Thus, while for a monopolist, independent goods pricing typically improves upon pure bundled prices by capturing additional sales from consumer types (for the case of a monopolist Firm A for example) in the upper left and lower right

[9]It might appear weakly dominant for each firm to at least try constructing a synthetic bundle by offering, say, $\tilde{p}_1^A = 3t/4$ instead of the clearly unacceptable $p_1^A = V^1$, hoping Firm B will do the same. While this would be profitable in the event that Firm B also offered an acceptable stand-alone price along with the proposed bundle price margin, t, such an offer would not be profitable if B offered, instead, the profile $(t, t, 3t/2)$ and so, the deviation is not weakly dominant. If weakly dominating deviations exist, I have not found them.

corners, for an imperfectly competitive firm facing a rival who only offers a bundle, these same agents are less willing to buy the independently priced good.

A qualitative empirical implication of the above argument is that mixed bundled pricing may be more common in monopolized multi-product markets than in imperfectly competitive markets. However, the next result illustrates that, in addition to the pure bundling equilibrium, there also can exist equilibria where both firms engage in mixed bundling. Thus, the qualitative claim stems primarily from the fact that multiple equilibria may exist in an imperfectly competitive market.

Theorem 4 characterizes necessary conditions for a symmetric equilibrium in non-trivial mixed bundle pricing and shows that these conditions are sufficient if $t_1 = t_2$.

Theorem 4. *Suppose $F_j(x) = x, j = 1, 2$. If $(\hat{p}_1, \hat{p}_2, \hat{P})$ is a symmetric equilibrium of the symmetric mixed bundle pricing game with positive sales of all products,*

$$\begin{aligned}
(\hat{p}_1 - c_1)(t_2 + \Delta) &= t_1 t_2 + 2(t_1 + t_2)\Delta - \Delta^2 - (\hat{p}_2 - c_2)(t_1 + \Delta) \\
\hat{p}_1 &= (t_1 - \Delta)(t_2 - \Delta) + c_1(t_2 - \Delta) + (P - c_2)(t_1 - \Delta) \\
\hat{p}_2 &= (t_1 - \Delta)(t_2 - \Delta) + c_2(t_1 - \Delta) + (P - c_1)(t_2 - \Delta)
\end{aligned}$$

where $\Delta = \hat{p}_1 + \hat{p}_2 - \hat{P}$. If $t_j = t$ then

$$\begin{aligned}
\hat{p}_j &= 11t/12 + c_j, \\
\hat{P} &= 8t/6 + c_1 + c_2
\end{aligned}$$

form a mixed bundling equilibrium of the game.

Proof. The proof of necessity uses the characterization of market segments in Lemma 1 to derive first order conditions, imposes symmetry and then derives the expressions. Matutes and Regibeau (1992) show that these conditions are sufficient in the case $t_1 = t_2$. See the Appendix for details. □

By invoking symmetry across firms, Theorem 4 offers a strategy for determining the mixed bundling equilibrium equilibrium when it exists. The first expression in the theorem is quadratic in Δ and it can be shown that only the positive root is a candidate for the equilibrium. For a given Δ, the second two equations are linear in $\hat{p}_j$ and can be solved for those values explicitly. For a given profile of costs and differentiation parameters, the price profile of, say, Firm B can be held at the candidate equilibrium prices and it can be checked directly if the same prices are global best responses for Firm A.

Lemma 1 illustrates the resulting market segmentation given the equilibrium above. Interestingly, the determination of the segments is independent of the costs of the two products. It depends solely on the difference between the sum of the independent goods and the equilibrium bundle price which in turn depends only on the (in this case) symmetric differentiation parameter t.

Theorems 3 and 4 demonstrate that there are markets where both mixed bundled pricing and pure bundled pricing can be equilibrium outcomes. The analysis indicates why. For it to be profitable for a firm to offer goods independently when its rival is offering a bundled price, the rival also must offer the other component good of the bundle. Pure bundling can arise because of an inability or reluctance of competing firms to coordinate in such a fashion.

This observation indicates that, in imperfectly competitive markets, the scope for price discrimination by one firm can depend on the pricing behavior of its rival. For the case of uniformly distributed consumers, as studied here, it is known that typically the optimal mechanism for a two-product monopolist discriminating against privately informed consumers, is a mechanism with strictly mixed bundle pricing: some consumers buy the monopolist's bundle, some buy only one of the two products, and some do not purchase at all. (See Manelli and Vincent (2006)). The ability of an imperfectly competitive firm in similar circumstances to use mixed bundling to price discriminate depends on its rival offering mixed bundled pricing as well. If the rival only offers pure bundle prices, then a price discriminating mixed bundle schedule by the other firm is not profitable.

4.1 Welfare Effects

If the market covered assumption holds in all cases, then total welfare relies solely on the partition of buyer types into the various bundle profiles, AA, AB, BA, BB. The profile associated with the symmetric equilibrium of independent goods pricing, maximizes the welfare available from product variety and is, therefore, optimal among all profiles considered here. Pure bundle prices is the worst symmetric outcome from the perspective of product variety while mixed bundle pricing is generally intermediate between the two. It offers some additional product variety over pure bundling but not as much as independent goods pricing. Since Theorem 2 demonstrates that independent goods pricing is not achievable, this social cost should perhaps be factored in when assessing the consequences of mergers of even firms that produce and sell unrelated products.

Firm profits are $(t_1+t_2)/2$ in the independent goods pricing mode. If $t_1 = t_2 = t$, Theorem 4 implies that firm profits are lower under the mixed bundling equilibrium ($67t/96$) and Theorem 3 shows profits are lower still with pure bundling ($t/2$). While implicit cooperation is required for firms to engage in price discrimination to improve on profits relative to the pure bundling equilibrium, ironically, the very ability to price discriminate via mixed bundling harms firm profits in an imperfectly competitive environment. Theorem 2 illustrates that when firms can offer mixed bundle profiles, they will do so rather than engage in individual goods pricing. However, the prices that arise when only individual goods pricing is feasible generate higher firm profits than in the mixed bundling equilibrium characterized in Theorem 4.

In the case where $t_1 = t_2$ consumer welfare effects are reversed when comparing individual goods pricing with the mixed bundling equilibrium. The individual goods prices are strictly lower in the latter and consumers can always choose that option under fully mixed bundlling, so their utility cannot be lower. The comparison is more ambiguous when comparing the mixed bundling equilibrium with pure bundling. Typically consumers are better off with the lower prices under pure bundling, however consumers in the far corner, for example at $(1, 0)$ gain $V^1 + V^2 - 22t/12$ under mixed bundling which is strictly better than the utility under pure bundling, $V^1 + V^2 - 2t$. For these consumers, the opportunity to mix and match outweighs the higher prices.

From an economic perspective, it is perhaps surprising that an imperfectly competitive market with independent goods has an equilibrium that duplicates the equilibrium that arises when goods are strong complements as in Matutes and Regibeau (1992). Mathematically, though, it is somewhat less surprising. Whenever the price profile is such that all consumers buy two products,the demand facing each firm is the same under either assumption on preferences and, therefore, best responses in this region are the same. This is because Firm A's product 1 becomes a market created complement with Firm B's product 2 when consumers are balancing whether to buy a bundle explicitly offered by each firm against buying a synthetic bundle constructed by the consumer. The different consequences of independent goods becomes more clear in the equilibrium outcome of pure bundling. Independent goods also create interesting differences in the case where a multi-product firm competes against two single product firms which is analyzed next.

5 Mixed Bundling in Blended Market Structures

What results if a multi-product firm faces two separate producers of competing products and engages in mixed bundling? In this structure, the pure bundle price outcome cannot emerge because Firms B1 and B2 are not offering a bundle price. Their best responses will generally be to offer prices for their individual goods even if the integrated rival does not offer its goods independently. Whether or not the integrated firm wishes to do so will depend on the distributions and the cannibalization effect on its bundle but, generally, it seems that it would wish to do so given that the other firms are offering their products. Thereom 2 still applies in this structure, though. The integrated firm will wish to offer mixed bundles prices even though the rival does not.[10]

Given that the integrated firm is engaging in non-trivial mixed bundling, the independent goods firms are at a disadvantage because of an induced form of double-marginalization. Even though their products are independent in consumption, when Firm A offers a bundle price for its two goods, a frontier emerges where some consumers are deciding between the synthetic bundle, BB and the actual bundle AA. Firm B1, for example, though, balances the margin it enjoys on the set of consumers $BA \cup BB$ against loss of sales on that competitive frontier. Firm B2 does a similar balance but on the set $AB \cup BB$. The failure of the independent firms to internalize the impact of their pricing on the profits of the other induces higher prices than would be offered by an integrated pair.

Equilibria in the blended market structure is generally difficult to characterize in any generality, however, if we assume $t_1 = t_2 = t$, the equilibria can be calculated computationally. The analysis is provided in the Appendix. The best responses for the individual firms are unique and symmetric. The best response for Firm A in its individual prices can also be derived explicitly. Then the derivative of Firm A's profit function with respect to the bundle price is plotted and shown to be decreasing everywhere in the mixed bundling region. Thus, its unique root, yields the best mixed bundling price. Table 1 provides the equilibrium profit margins for this case and, for comparison, provides the similar equilibrium results from Theorem 4 for the case of competing multi-product firms. Thus, the table provides a snapshot of the price

[10] The structure discussed in this section is similar to that analyzed in Gans and King (2006) Proposition 4, though, again they assume a sort of bundling commitment. To my knowledge, they are the first to observe that bundling by one firm induces complementarity across otherwise independent goods.

effects that can arise when two individual independent product firms that initially compete against a mixed bundling multi-product firm merge and then the merged entity competes in mixed bundled pricing against the incumbent rival.

Table 1: Equilibrium Comparisons ($t_1 = t_2 = .5$)

Variable	Individual Pricing (Four Firms)	Blended Market (Three Firms)	Mixed Bundling (Duopoly)	Pure Bundling (Duopoly)
$\tilde{p}_1^A$	.5	.607	$11/24 = .458$	.25
$\tilde{p}_1^B$	.5	.44	$11/24 = .458$	.25
$\tilde{p}_1^A + \tilde{p}_2^B$	1	1.11	$11/12 = .92$	.5
$\tilde{P}^A$	1	.81	.67	.5
Π^A	.5	.487	$67/192 = .35$	.25
$\Pi^{B1} + \Pi^{B2}$	.5	.395	$67/192 = .35$	.25

Table 1 illustrates some intriguing equilibrium dilemmas. Suppose that there are initially four single product firms, all of which can only price the products individually. Now, suppose that only the product A firms merge and acquires the capability and incentive to engage in mixed bundling. Theorem 2 implies that new merged entity must change its prices and engage in non-trivial mixed bundling. This incentive tends to induce Firm A to raise the price of its stand-alone product and lower the price of the two-good bundle in this case to $p_j^A = .607$ and $P^A = .81$ respectively. From Firm B2's perspective, the pricing behavior of its rival changes the nature of competition. Initially, competing only against Firm A's product 2, the rise in price of p_2^A would induce it to raise its price as well since single-product Hotelling goods are strategic complements. In the uniform case with $t = .5$, Firm B2's best response would be to raise its price to .55. However, with mixed bundling, products B2 and A1 become complements in competing against Firm A's bundle. The fall in price of the bundle and the rise in Firm A's price of product 1 induces Firm B2 to *lower* its price in response. Products that were strategically independent become strategic substitutes. Although product A1 and B2 are intrinsically independent, mixed bundling induces a competitive complementarity. Table 1 shows that the equilibrium consequence of this change is to lower the profits of both firms.

If Firms B1 and B2 acquire the capability to mixed bundle, then one possibility could be for it also to engage in mixed bundling and generate the slightly lower profits of .35. Another possible outcome could be for both firms to revert to the pure bundling outcome that generates the lowest possible profits for both firms. In either case, counter-merger by the B firms ultimately results in lower equilibrium profits for both firms. Note that computations indicate that the asymmetric outcome where a merged Firm B continues to price its products individually against the mixed bundle prices of Firm A when it has the capability of bundle pricing does not form a best response presumably because of the logic underlying Theorem 2.

In the example illustrated in Table 1, total profits are highest when firms are independent and lowest when they are both integrated. Thus, the ability to acquire mixed bundling capabilities do not provide an incentive for firms to merge either unilaterally or as a counter-response. Still, a merger of the independent firms can arise for reasons other than for pricing purposes. Nevertheless, one consequence of the merger will be to generate the mixed bundling strategic mode. Consider the effects of a merger of the B firms. Assuming that a mixed bundle pricing equilibrium emerges after a counter-merger, the price effects of a merger of the independent firms are informative. The total price of each bundle falls. The BB price falls because of the elimination of double-marginalization and the price of AA falls because these two products are strategic complements. The stand-alone prices of the newly integrated Firm B's goods rise to limit the cannibalization of its newly formed bundle. However, the stand-alone price of Firm A's products falls significantly. One reason is because the cannibalization effect for this firm is now mitigated with the fall in its bundle price. However, another important factor is that the rise in Firm B's stand-alone prices induces a fall Firm A's stand-alone prices because, as noted before, with mixed bundle pricing, products B1 and A2 are now strategic substitutes. Presumably price effects of counter-mergers of this sort will vary with different market characteristics.

6 Conclusion

Not all multi-product firms have an incentive to engage in mixed bundling. It is unlikely that when General Electric owned both NBC and General Electric Aviation it had any incentive to bundle prices for commercials on Saturday Night Live with jet engines. However, when the same consumers can potentially buy the different products, even when they are independent, the scope and incentive for price discrimination arises and thus for mixed bundling.

The environment examined in this paper is special. Attention is restricted to two firms with two products and to a market where, because of the Hotelling feature, total sales are assumed constant. Furthermore, most of the positive equilibrium results require assuming uniformly distributed consumers. Nevertheless, I believe the model offers some important insights that are likely to extend to broader environments.

Theorem 2 holds for any pair of independent distributions possessing a density – the McAfee, McMillan and Whinston intuition that mixed bundling is generally strictly more profitable extends to imperfectly competitive, multi- product firms. If products are differentiated, this behavior imposes potential welfare costs as the resulting pricing induces consumers to select into a suboptimal allocation of variety. Since mixed bundled pricing is, in theory, at least, available to any multi-product firm, this effect should be an additional consequence considered when assessing the implications of mergers, even of horizontally unconnected firms. Generally, so-called conglomerate mergers, mergers of firms that do not directly compete, have been viewed with relaxed scrutiny [11]. The results here suggest that, in some cases, a welfare loss can arise from such mergers.

The fact that both pure bundling and mixed bundling equilibria can exist in the same market is shown here concretely for a specific environment. However, the qualitative explanation for why a pure bundling equilibrium can persist even when mixed bundling is feasible is likely general. Mixed bundling may only be successful against a rival's bundle in these markets if the rival itself offers the second part of bundle that the consumer is expected to construct for herself. This requires implicit cooperation among non-cooperatively pricing firms. Interestingly, in these examples, if firms are successful in arriving at the mixed bundle price equilibrium, efficiency and firm profits rise but consumer welfare in aggregate falls. This particular welfare result, however, is likely to be fragile and, I conjecture, would change with different models of product differentiation and different assumptions on the distribution of consumer types. For example, the total surplus conclusion is largely an artifact of the Hotelling model with full market coverage. If reductions in price were to cause market expansion as well as market stealing, the conclusion that bundle pricing reduce total welfare would likely be mitigated.

Insights from the blended market model are also likely to extend beyond the special case. If two independent firms compete in different markets against the same multi-product firm, then even though their products are not intrinsically complements, they become complements through the mixed bundled pricing of the rival. This puts the independent firms at a disadvantage in the market place because of what might otherwise have been an unexpected source of double- marginalization.

Multi-product competition with differentiated products generates profit functions

[11] See, for example, Kolasky, 'After fifteen years of painful experience ....,the U.S. antitrust agencies concluded that antitrust should rarely, if ever, interfere with any conglomerate merger'. P. 1.

that are intrinsically neither supermodular nor quasi-concave. Thus, our standard tools for equilibrium analysis – lattice-based and topological fixed point theorems – are not that usable in these models. The importance and ubiquity of multi-product competition points to the need for additional tools to aid equilibrium analysis.

7 Appendix

7.1 Proof of Theorem 1

Fix Firm B's bundle price at the candidate equilibrium, $\hat{P}^B = t_1+c_1+c_2$. A consumer at $(0,0)$ who purchases Firm B's bundle obtains utility

$$V^1 + V^2 - 2t_1 - t_2 - c_1 - c_2 \geq\geq 0$$

by (2). Therefore, at any price P^A Firm A sales are given by the area of the intersection of the half-plane $x_2 \leq (t_1 + t_2 - (P^A - \hat{P}^B) - 2t_1x_1)/(2t_2)$ and the unit square.

Define $\Delta = P^A - \hat{P}^B < t_1 + t_2$. The partition could occur in three ways: a) the manifold intersect the top and bottom of the square, ($\Delta \in [-(t_1 - t_2), (t_1 - t_2)]$); b) it could intersect the left edge and the bottom of the square ($\Delta > (t_1 - t_2)$) or; c) the top of the square and the right edge of the square ($\Delta < -(t_1 - t_2)$). Let $\mu(E(\Delta))$ denote the measure of Firm A's bundle sales when the price difference is Δ.

In Case a), $\mu(E(\Delta)) = (t_1 - \Delta)/(2t_1)$ and $\partial\mu(E(\Delta))/\partial P^A = -1/(2t_1)$. The derivative of Firm A's profit is, thus,

$$(t_1 + \hat{P}^B - P^A)/(2t_1) - (P^A - c_1 - c_2)/(2t_1) = 2(t_1 + c_1 + c_2) - 2P^A$$

which is greater than zero for $P^A < \hat{P}^B$ and less than zero otherwise. Thus, within this region, $P^A = \hat{P}^B$ is a best response.

In Case b), $\mu(E(\Delta)) = (t_1 + t_2 - \Delta)^2/(8t_1t_2)$ and $\partial\mu(E(P^A))/\partial P^A = -(t_1 + t_2 - \Delta)/(4t_1t_2) < 0$. Therefore, in this region the derivative of Firm A's profit satisfies

$$\begin{aligned}(\mu(E) + (P^A - c_1 - c_2)\partial\mu(E(P^A))/\partial P^A &= (t_1 + t_2 - \Delta - 2(P^A - c_1 - c_2))(t_1 + t_2 - \Delta)/(8t_1t_2) \\ &\leq (2t_2 - 2(2t_1 - t_2))(t_1 + t_2 - \Delta)/(8t_1t_2) \\ &= 4(t_2 - t_1)(t_1 + t_2 - \Delta)/(8t_1t_2) \\ &< 0.\end{aligned}$$

The inequality comes $\Delta > t_1 - t_2$ and $P^A > \hat{P}^B + t_1 - t_2 - c_1 - c_2 = 2t_1 - t_2$. Since Firm A profits are decreasing in P^A everywhere in this region, there can be no best response in region b).

In Case c), $\mu(E(\Delta)) \geq \mu(E(-(t_1 - t_2)) = (2t_1 - t_2)/(2t_1)$ and $\partial\mu(E(\Delta))/\partial P^A \geq \partial\mu(E(-(t_1 - t_2)))/\partial P^A = -1/(2t_1)$ where the partial derivative is evaluated at the price at the boundary between region a) and c). Therefore, in this region the derivative of Firm A's profit satisfies

$$\begin{aligned}\mu(E) + (P^A - c_1 - c_2)\partial\mu(E)/\partial P^A &\geq (2t_1 - t_2 - (P^A - c_1 - c_2))/(2t_1)\\ &\geq (2t_1 - t_2 - (\hat{P}^B - (t_1 - t_2) - c_1 - c_2))/(2t_1)\\ &> (2(t_1 - t_2))/(2t_1).\end{aligned}$$

This is positive and therefore the best response cannot lie in this region,

7.2 Proof of Theorem 2

Proof. If firms can only select individual goods prices then each product market is the standard Hotelling market and, for the general, symmetric distributions, any symmetric equilibrium in prices must satisfy

$$p_j^A = p_j^B = \hat{p}_j = \frac{t_j}{f_j(1/2)} + c_j, j = 1, 2. \tag{4}$$

It is well-known that if the distribution of consumers is uniform, then (4) is also sufficient but a broader class of distributions would generate similar conditions.

Now suppose that the strategic mode is mixed bundling and that prices $(\hat{p}_1, \hat{p}_2, \hat{p}_1 + \hat{p}_2)$ are offered by both firms. Any consumer who buys both goods from the same firm, pays the sum of the independent goods prices. At these prices, total sales, profits and consumer utility are the same as in the equilibrium of the independent goods pricing game. It is now shown that $(\hat{p}_1, \hat{p}_2, \hat{p}_1 + \hat{p}_2)$ is not a mutual best response.

Suppose Firm A offers the price vector, $(\hat{p}_1 + \Delta, \hat{p}_2, \hat{p}_1 + \hat{p}_2), \Delta \geq 0$. Holding Firm B prices fixed at $(\hat{p}_1, \hat{p}_2, \hat{p}_1 + \hat{p}_2)$, consumers' purchases are determined from Lemma 1 with $\bar{x}_1 = \underline{x}_2 = 1/2$ and $\bar{x}_2 = \frac{t_2+\Delta}{2t_2}, \underline{x}_2 = \frac{t_1-\Delta}{2t_1}$. (See Figure 1)

Profits from such a price profile for Firm 1 are

$$\Pi(\Delta) = (\hat{p}_1 - c_1 + \hat{p}_2 - c_2)\mu(AA) + (\hat{p}_1 + \Delta - c_1)\mu(AB) + (\hat{p}_2 - c_2)\mu(BA).$$

At $\Delta = 0$, profits are equal to the profits generated by the candidate independent goods price profile, $(\hat{p}_1, \hat{p}_2)$. Differentiating with respect to Δ (the final term is independent of Δ) and evaluating at $\Delta = 0$ yields

$$\begin{aligned}\frac{\partial \Pi}{\partial \Delta}_{\Delta=0} &= (\hat{p}_2 - c_2)F_1(1/2)f_2(1/2)/(2t_2) + (1 - F_2(1/2))(F_1(1/2) - (\hat{p}_1 - c_1)f_1(1/2)/(2t_1)) \\ &= (\hat{p}_2 - c_2)F_1(1/2)f_2(1/2)/(2t_2) \\ &> 0.\end{aligned}$$

where the equality follows from the property of $\hat{p}_1$ as a best response (4) and the strict inequality follows from the assumption that both goods yield strictly positive profits. □

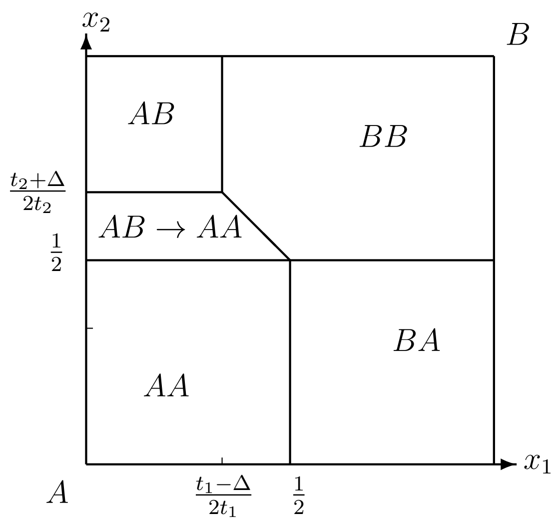

Figure 3: Proof of Theorem 2

7.3 Proof of Lemma 2

Proof. For concreteness, let $j = 1$. If $p_2^B > V^2$, no consumer will purchase Firm B's product 2. The set of consumers indifferent between the two bundles is given by

$$x_2 = (t_1 + t_2 + P^B - P^A - 2t_1)/(2t_2).$$

Along this manifold, consumers obtain positive utility if $2(V^1+V^2)-t_1-t_2 \geq P^A+P^B$.

A consumer will purchase Firm A's product 1 alone only if $p_1^A < V^1$ and if this purchase dominates buying the bundle from either Firm A or B. The first condition is

$$V^1 - t_1x_1 - p_1^A \geq V^1 + V^2 - t_1x_1 - t_2x_2 - P^A$$

or

$$x_2 \geq \frac{V^2 - (P^A - p_1^A)}{t_2}.$$

The consumer prefers A's product 1 alone to Firm B's bundle if and only if

$$V^1 - t_1 x_1 - p_1^A \geq V^1 + V^2 - t_1(1 - x_1) - t_2(1 - x_2) - P^B$$

or

$$2x_2 \leq t_1 + t_2 + P^B - p_1^A - V^2 - 2t_1 x_1.$$

The intersection of these two half-spaces with the set $[0,1]^2$ are the consumers that prefer buying product 1 alone from A to either bundle. The intersection of these half-spaces with the manifold where consumers are indifferent between the two bundles is the point

$$\left(\frac{t_1 + t_2 + P^A + P^B - 2V^2 - 2p_1^A}{2t_1}, \frac{V^2 - (P^A - p_1^A)}{t_2}\right).$$

The set of consumers who buy product 1 alone is positive if and only if this point lies in $[0,1]^2$ which holds if and only if (3) is satisfied. □

7.4 Proof of Theorem 3

Proof. Suppose that Firm B offers $(V^1, V^2, t + 2c)$. Consider Firm A responding to Firm B's prices with some bundle price, P^A, and either a price for (say) good 1, p_1^A such that some consumers purchase good 1 alone, or a price, V^1 such that no consumers buy good 1 alone. It is first shown that the second option dominates the first for any bundle price P^A. (The symmetric argument holds for good 2). This establishes that an essentially pure bundle price is a best response to Firm B's essentially pure bundle price. Theorem 1 shows that $P^A = t + 2c$ is the best response in pure bundle pricing.

Suppose Firm A offers $(p_1^A, p_2^A, P^A), P^A > P^B = t + 2c$. Consumers who prefer Firm A's bundle to Firm B's bundle are in the set

$$x_2 \leq 1 - \frac{P^A - P^B}{2t} - x_1.$$

This manifold has vertical intercept less than one so, by Lemma 2, a consumer prefers good 1 independently from Firm A over Firm A's bundle only if

$$p_1 \leq \frac{P^A + P^B}{2} + t - V^2.$$

and consumers will purchase good 1 alone only if

$$x_2 \leq (3t + 2c - p_1^A - V^2)/t - 2x_1, x_2 \geq (V^2 - (P^A - p_1^A)/t.$$

Figure 4 illustrates the set of consumers who respond to the offers $(p_1^A, p_2^B, P^A), (V^1, V^2, t + 2c)$ by purchasing good 1 alone. The set can be partitioned in two regions, E and F. The region E represents consumer types who would have purchased Firm A's bundle if instead the price offers $((V^1, V^2, P^A), (V^1, V^2, t + 2c))$ were made (Firm A responded to Firm B's offer with only a serious bundle price, P^A) and F is the region of consumers who buy product 1 alone but who would have responded to Firm A's pure bundle offer by purchasing from Firm B. The uniform distribution and geometry implies that the mass of $E, \mu(E)$, is no less than the mass of F consumers, $\mu(F)$, (and is equal only if $p_1^A > t + P^B - V^2$. This condition ensures that the manifold separating purchasers of product A1 from purchasers of Firm B's bundle intersects the vertical axis insider the unit square. In this case, the area of the two triangles, E and F are the same.)

Compared to the price profile $((V^1, V^2, P^A), (V^1, V^2, t + 2c))$, Firm A experiences a net loss of $P^A - 2c - (p_1^A - c) = P^A - p_1^A - c$ in E and a net gain of $p_1^A - c$ in F. Thus, compared to the pure bundle pricing option at price P^A, the revenue effect of inducing good 1 purchases is

$$-(P^A - p_1^A - c)\mu(E) + (p_1^A - c)\mu(F) \leq (2p_1^A - P^A)\mu(E)$$

Since $p_1 \leq \frac{P^A + P^B}{2} + t - V^2$ in order for some consumers to buy product 1, the profit impact of independent goods pricing is bounded above by

$$\mu(E)(P^A + P^B + 2t - 2V^2 - P^A) = \mu(E)(3t + 2c - 2V^2).$$

Under the hypothesis of the Proposition, this is less than zero and offering an essential pure bundle price P^A is a better option for Firm A than inducing independent purchases of good 1.

A similar argument holds for the case, $P^A < P^B$. In this case, the binding constraint for Firm A to make positive sales of product 1 from Lemma 2 is that $2p_1^A \leq 2P^A + 2t - 2V^2$ so

$$2p_1^A - P^A \leq P^A + 2t - 2V^2 \leq P^B + 2t - 2V^2 = 3t + 2c - 2V^2.$$

It is immediate with these prices that $\mu(E) > \mu(F)$ and, once again, the increase in Firm A profits from making sales of its product 1 alone are bounded by $\mu(E)(3t +$

$2c - 2V^2) < 0$. Since for all bundle prices, P^A, offering a price profile that yield no individual good sales dominates any profile with such sales, and since $P^A = t + 2c$ is the pure bundling best response to $P^B = t + 2c$, essentially pure bundling prices remains an equilibrium even if mixed bundling is available. □

7.5 Proof of Theorem 4

For a fixed vector of prices, $(p_1^A, p_2^A, P^A, p_1^B, p_2^B, P^B)$, if we define per product profit margins as

$$\tilde{p}_j^i = p_j^i - c_j, \tilde{P}^i = P^i - c_1 - c_2,$$

then the definitions of $\bar{x}_j, \underline{x}_j$ are equivalently defined substituting per product profit margins in place of prices and, assuming all consumers buy all bundles, the market is partitioned as given in Lemma 1. The uniform distribution then gives the measures of the four market segments as the area of the sets AB, AA, BB, BA:

$$\mu(AB) = (1 - \bar{x}_2)\underline{x}_1; \mu(BA) = (1 - \bar{x}_1)\underline{x}_2; \mu(AA) = \bar{x}_2\bar{x}_1 - (\bar{x}_2 - \underline{x}_2)(\bar{x}_1 - \underline{x}_1)/2.$$

These definitions yield the following:

$$\begin{aligned}
\frac{\partial\mu(AB)}{\partial\tilde{p}_1^A} &= -\frac{(1-\bar{x}_2)}{2t_1} - \frac{\underline{x}_1}{2t_2} \\
\frac{\partial\mu(AB)}{\partial\tilde{P}^A} &= \frac{\underline{x}_1}{2t_2} \\
\frac{\partial\mu(AA)}{\partial\tilde{p}_1^A} &= \frac{\underline{x}_1}{2t_2} \\
\frac{\partial\mu(AA)}{\partial\tilde{P}^A} &= -\frac{\bar{x}_1}{2t_2} - \frac{\bar{x}_2}{2t_1} + \frac{\bar{x}_1 - \underline{x}_1}{4t_2} + \frac{\bar{x}_2 - \underline{x}_2}{4t_1} \\
&= -\frac{t_1(\bar{x}_1 + \underline{x}_1) + t_2(\bar{x}_2 + \underline{x}_2)}{4t_1t_2}.
\end{aligned}$$

Using the fact that

$$t_j(\bar{x}_j + \underline{x}_j) = (2t_j + P^B - P^A + p_j^B + p_i^A - p_j^B - p_i^B)/2,$$

we can also write

$$\frac{\partial\mu(AA)}{\partial\tilde{P}^A} = -\frac{t_1 + t_2 + \tilde{P}^B - \tilde{P}^A}{4t_1t_2}.$$

Firm A's profit function is

$$\Pi(\tilde{p}_1^A, \tilde{p}_2^A, \tilde{P}^A) = \tilde{p}_1^A\mu(AB) + \tilde{P}^A\mu(AA) + \tilde{p}_2^A\mu(BA). \tag{5}$$

Thus the first order condition for $\tilde{p}_1^A$ is

$$0 = \mu(AB) - \tilde{p}_1^A \frac{1 - \bar{x}_2}{2t_1} + (\tilde{P}^A - \tilde{p}_1^A)\frac{\underline{x}_1}{2t_2} \tag{6}$$

A symmetric condition holds for $\tilde{p}_2^A$. The first order condition for $\tilde{P}^A$ is

$$0 = \mu(AA) + \tilde{p}_1^A \frac{\underline{x}_1}{2t_2} - \tilde{P}^A \frac{t_1 + t_2 + \tilde{P}^B - \tilde{P}^A}{4t_1 t_2} + \tilde{p}_2^A \frac{\underline{x}_2}{2t_1} \tag{7}$$

Assuming a symmetric solution, $\tilde{p}_j^A = \tilde{p}_j^B, \tilde{P}^A = \tilde{P}^B, \underline{x}_j = 1 - \bar{x}_j$, $\mu(AB) = \mu(BA)$ and $\Delta = \tilde{p}_1^A + \tilde{p}_2^A - \tilde{P}^A$. Summing the first order conditions for $\tilde{p}_1^A, \tilde{p}_2^A$ then eliminates the $\tilde{p}_j^A$s yielding an expression only in terms of Δ:

$$(t_2 - \Delta) + (t_1 - \Delta) = 2(t_2 - \Delta)(t_1 - \Delta).$$

This is a convex quadratic in Δ (and therefore p_j^A) so necessary second order conditions imply selecting the smallest of the roots. Substituting into (6) for $\tilde{p}_1^A$ and (7) for $\tilde{P}^A$, yields a necessary conditions for a symmetric equilibrium. If $t_1 = t_2 = t$, then the solution for $\Delta = t/2$. This then gives $p_j = 11t/12 + c_j, P = 8t/6 + c_1 + c_2, \underline{x}_j = (1 - \bar{x}_{j'}) = 1/4$ and firm profits are $(2/3 + 1/32)t = .698t$.

(Sufficiency) Suppose $t_j = t$. For a given $\tilde{p}_2^B, \tilde{P}^B, \tilde{P}^A$, in order to remain a non-trivial mixed bundling solution we must have $\tilde{P}^A \leq \tilde{p}_1^A + \tilde{p}_2^A$ and $\tilde{P}^A \geq \tilde{p}_j^A + \tilde{p}_{j'}^B - t$ (the first expression is where mixed bundling converges to individual goods pricing and the second ensures that the sets AB, BA are non-empty). Along with the requirement that prices be low enough to ensure positive sales, this is a compact set and since (5) is continuous and smooth, a solution exists and if the solution is in the interior, the first order conditions will be satisfied. We show that the first order conditions (6) rule out a solution at the boundaries of the mixed bundling set and then show that the derived necessary conditions are the only conditions that satisfy conditions for a local maximum in the interior of the set of mixed bundle prices. The solution is then shown to dominate the solution when the Firm only offers a bundle.

The derivative of Firm A profit with respect to $\tilde{p}_1^A$ (equation (6)) is convex and quadratic in $\tilde{p}_1^A$. Therefore, there can be at most one optimal interior solution, and the only other possible solution is at an upper bound of $\tilde{p}_1^A$ where no sales are made of product 1 (AB is empty). If $\mu(AB) = 0$ because $\underline{x}_1$ vanishes (this occurs only if $P^A \geq P^B$), then equation (6) is strictly negative as p_1^A approaches this upper bound and cannot be a solution. If $\mu(AB) = 0$ because $1 - \bar{x}_2$ vanishes (this occurs only if $P^A \leq P^B$ and as $\tilde{P}^A - \tilde{p}_1^A$ approaches $\tilde{p}_2^B - t$ from above) then equation (6))approaches

$$(\tilde{P}^A - \tilde{p}_1^A)\frac{\underline{x}_1}{2t_2}.$$

But the candidate equilibrium profile sets $\tilde{p}_2^B = 11t/12$ so as $\tilde{p}_1^A$ approaches its upper limit, $\tilde{P}^A - \tilde{p}_1^A$ is negative and and this boundary cannot be a solution. Thus, if a mixed bundle solution is optimal, it must be an interior solution where the sets AB and BA are non-empty. Symmetry in the t_js and p_j^{B}' implies that the two sets are mirror symmetric in the 45 degree line. This argument implies that when optimizing with respect to P^A within the class of mixed bundling solutions, the p_j^As will be selected to be equal to each other and solve the lower of the two roots of (6). The derivative of profit with respect to P^A is concave and quadratic and, since the solution cannot be at a boundary, the solution must be in the interior at the higher of the two roots. Thus the optimal solution within the class of mixed bundling solutions is that given in the Theorem.

Firm A has the option, if it chooses, to price its individual products so high that only its bundle is sold. Price profiles of this sort yield pure bundling solutions at best for Firm A (if the market profile is similar to P3 in Figure 4 it is worse for Firm A than P2 so if we can show pure bundling is not as profitable as mixed bundling, we know this outcome is also not as good.) If $\tilde{P}^A > \tilde{P}^B$, then Firm A profits are always decreasing in $\tilde{P}^A$, so this region is never optimal. If $\tilde{P}^A \leq \tilde{P}^B$, then Firm A sales are are given by

$$(8t^2 - (2t - \tilde{P}^B + \tilde{P}^A)^2)/(8t^2).$$

Using $\tilde{P}^B = 8t/6$ then the derivative of Firm A's profit in this region is the concave quadratic

$$-3(\tilde{P}^A)^2 - \frac{8t}{3}\tilde{P}^A + \frac{68t^2}{9}.$$

The higher root of this expression is

$$\tilde{P}^A = 4t(\sqrt{55} - 2)/18.$$

This yields a profit of $.6773t$ which is lower than the strictly mixed bundling profit.

Conceivably, Firm A could offer a profile of prices such that BA is empty but AB is not. Note that, given the equilibrium P^B, p_2^B, any optimal p_1^A must generate a value of $\underline{x}_1 < 1/2$. If not, then (6 would be strictly positive. This along with symmetry implies that if Firm A also offered p_2^A at the same price, the sets AB and BA would be symmetric and would not intersect. However, holding the A's bundle price and Firm B's prices fixed, if posting a price such that AB sales are made is better than a price such that no product 1 sales are made and just selling the bundle, the same must be true of posting a price for p_2^A such that $\mu(BA) = \mu(AB)$ sales of product 2 are made. But this is a mixed bundling solution and the best solution of this type satisfies the necessary conditions of the theorem.

7.6 Mixed Bundling in Mixed Markets

In this market structure, Firm A markets both products 1 and 2 and engages in mixed bundle pricing. However, the differentiated products from site B are produced and priced by two independent firms. Each firm, B1, B2, sets its price, p_1^B, p_2^B independently and to maximize its profits. The market partitions are as outlined in Lemma 1, however, since the price of bundle BB is the sum of the standalone prices, p_j^B, the definitions of the boundaries between BB and AB or BA are simplified somewhat to

$$\underline{x}_j = (t_1 - (p_j^A - p_j^B))/(2t_j).$$

For convenience, note that

$$\bar{x}_j - \underline{x}_j = (p_1^A + p_2^A - P^A)/(2t_j).$$

Given the uniform distribution, the measures of the sets are computed in the same way as for the mixed bundling two-firm market. Now, however, the sales of (say) product B1, then is the sum $\mu(BA) + \mu(BB)$. Firm B1 then selects its best response in price, p_1^B holding fixed the three prices of Firm A and the price of Firm B2. The first order condition yields a unique solution

$$0 = \mu(AB) + \mu(BB) - \tilde{p}_1^B/(2t_1).$$

A similar condition, replacing 1 with 2 and BA for AB, holds for the other independent firm. These equations are linear decreasing in $\tilde{p}_j^B$ and therefore there is a unique solution. The derivatives of the profit function for the integrated firm are the same as given in (6) and (7). As before, (6) is convex, quadratic in $\tilde{p}_j^A$ so the lowest root is the solution. Symmetry in t implies that if $\tilde{p}_1^B = \tilde{p}_2^B$, Firm A selects $\tilde{p}_1^A = \tilde{p}_2^A$. These solutions are then inserted in (7). A Mathematica plot shows that (7) is decreasing everywhere in the range for P^A and the solution is derived computationally. Firm A can also choose to pure bundle against the candidate prices by choosing very high individual prices and a bundle price so low that Firm B products are sold only as a bundle as well however this yields a solution that is dominated by mixed bundling.

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