Activity Rules and Equilibria in the Combinatorial Clock Auction

1. Introduction

For the past 20 years, auctions have become a widely used tool in allocating broadband spectrum. These auctions help efficiently allocate the spectrum and encourage competition among mobile phone providers. Combinatorial auctions, auctions where bidders are allowed submit all-or-nothing bids for groups of goods, are becoming an increasingly popular way of auctioning the spectrum. Combinatorial auctions are especially in environments where the auctioneer expects there to be lots of complementarities between goods. The combinatorial clock auction is a relatively new combinatorial auction design, which has been implemented in many countries, including the United Kingdom and the Netherlands (Cramton 2009). This design, initially proposed in “The Clock-Proxy Auction” by Ausubel, Cramton and Milgrom (2006), is believed to have a number of advantages over the simultaneous multiple round auction, the auction design which has been widely used to allocate the spectrum in the past. This paper considers how best to constrain activity in the combinatorial clock auction. It also considers whether an equilibria exists where the final allocation is efficient and all bidders bid sincerely in all rounds under either of the two common activity rules.

The combinatorial clock auction is a two stage auction that begins with a clock auction, and ends with a round where bidders can submit supplementary bids. The items being auctioned are divided into categories of homogeneous goods. In each round of the clock phase, the auctioneer announces prices for each category, raising the price for any category where demand exceeded supply in the previous round. Bidders then respond by publicly submitting quantities demanded at those prices. These bids are treated as package bids, bidders can either win everything or nothing that they bid on in that round. The clock phase continues until supply exceeds demand in all categories. This clock phase is a straightforward design that hopefully allows bidders to learn information about prices. If bidders have not fully determined their values for each package or if player’s valuations are not independent, the clock phase enables bidders to learn more information about other player’s valuations and update their own. After the clock phase ends bidders can submit an additional set of package bids. Each bidder is allocated at most one package, and the final allocation is chosen to be the allocation which maximizes the sum of bidder’s as-bid values.

What is of particular interest in this auction design is how bids should be constrained to encourage activity in every round. Without something to force the bidders to bid, bidders would have strong incentives to wait until it seemed like the auction was ending before bidding. This behavior can be discouraged in a clock auction through a rule that constrains all future bids based on previous bid. Activity rules can be used to force bidders into submit meaningful bids in each round of the auction. A variety of activity rules have been proposed for this auction. Many of the current rules are point based, restricting bidders to bidding on packages that are not "larger" in terms of point total then the package bid on in the previous round. Ausubel, Cramton, and Milgrom (2006) proposed a rule that uses the law of revealed preference, and a variety of rules have been proposed that combine the two techniques (for instance Ofcom 2011, or Ausubel, Cramton 2011). This paper considers how point based and revealed preference rules can influence how bidders behave.

One of the issues with a point based activity rule is that when a bidder bids his demand in early rounds, it can constrain him from bidding his demand in later rounds. This issue seems to be “resolved” in most practical implementations of these dynamic auctions by calibrating the points to reflect something that the regulator believes is a good indication for demand, for instance the population of the region the spectrum corresponds to. But this paper demonstrates that there exist fairly normal preferences for which no set of points will prevent bidders from being constrained from bidding their demand by the activity rule. This makes bidding in the auction more complicated for the bidders, and limits the ability to extract meaningful information from the clock phase.

One of the most important benefits of a dynamic auction is the potential for price discovery (Cramton 2009). Unlike in the theoretical modeling, bidders often have not developed a completely accurate set of valuations for every package. With the large number of packages that bidders can bid on, an auction mechanism which allows bidders to focus on a much smaller subset of the possible packages can really help make the bidder's valuation problem more manageable. So an auction mechanism which encourages bidders to bid sincerely in every round provides every bidder with very useful information. This paper shows that activity rules like an eligibility point rule provide incentives for bidders to distort their bids in order to maintain high levels of eligibility, really hindering the potential for price discovery.

This paper first describes the auction design, highlighting the important differences between the combinatorial clock auction and the simultaneous multiple round auction, the design that has been used in most spectrum auctions. It then considers an equilibrium in the auction where goods are homogeneous and bidders have non-increasing marginal value. In this equilibrium bidders bid sincerely and the final allocation is efficient. Finally it discusses point based and revealed preference activity rules in the heterogeneous case, and how those activity rules can affect the existence of equilibria where bidders bid sincerely and the outcome is efficient.

2. The Combinatorial Clock Auction

This auction has three important differences from the simultaneous multiple round auction. The simultaneous multiple round auction is the auction format which has been used in most spectrum auctions. In that auction, all items are auctioned simultaneously. In each round, bidders submit bids for each item they want, with some set of bids being selected as “provisional winning” bids at the end of each round. Bidders cannot withdraw from bidding on provisionally winning bids unless some other bidder bids on them in a future round. This procedure continues until there are no bids for any of the items. In auctions with lots of complementarities across goods, it has been postulated that the combinatorial clock auction has a number of advantages over this design. These advantages are due to three important aspects of the auction design: the ability for bidders to submit package bids, the pricing rule, and the activity rule.

i. Package Bidding

The combinatorial clock auction allows package bidding, while the simultaneous multiple round auction does not. This has many advantages, particularly when some bidders view goods as complements. Without package bidding, when bidders are bidding on complements they face an exposure problem. Bidders can lose essential items, but still win, and have to pay for, other items which are far less useful without their complements. This can occur in the simultaneous multiple round auction due to "provisionally winning bids."

For instance, if a bidder wanted a package AB, and was declared a provisional winner for A, but then the price of B rose to the point where he no longer wanted to win AB, he could be forced to win A at a loss. While this cannot happen when goods are substitutes, since increases in the price of B rises would not decrease the bidders demand for A, it can occur in environments where some goods are complements. This encourages bidders to reduce greatly reduce their demand, even to the point of not bidding for items they value highly. This leads to reduced revenue in the auction, and inefficiencies in the final allocation. Package bidding removes any risk of exposure, so bidders can express their demand without risking paying for a group of items that they do not want to win.

ii. The Pricing Rule

The Combinatorial Clock Auction is a core selecting auction. Using the final set of bids, it selects a final allocation and prices that lie in the core, the set of payoffs such that no group of bidders would be willing to offer the seller more than he is receiving now. This auction selects the bidder-optimal point in the core which is closest to the Vickery-Clark-Groves (VCG) prices, the mechanism which makes bidding your true value weakly dominant. When all bidders have substitute preferences, the VCG prices are the unique bidder-optimal point in the core.

The simultaneous multiple round auction, unlike the combinatorial clock auction, has a uniform pricing rule. For each item, bidders pay the clearing price. A uniform pricing rule creates incentives for demand reduction, which can lead to inefficient outcomes, reduce revenue and make it harder for bidders to discover prices. For instance, consider the case where two homogeneous goods are being auction to two bidders with non-increasing marginal values. While bidders could bid up to their value, they could also reduce their demand to just one good in the first round, each winning one item and paying zero. Bidders have given away no information about their values, the auctioneer is giving away valuable goods, and in the case where one bidder has a higher marginal value for both items this allocation is inefficient.

Unlike uniform prices, a VCG mechanism encourages sincere bidding. Bidders pay the opportunity cost of awarding the good to them, so their payoff is exactly the surplus they bring to the auction, and then have no incentive to reduce demand. To find this, remove the bidder from the auction. Then calculate the new value maximizing assignment. The bidder is awarded the difference of the as-bid value of that assignment and the as-bid value of the actual assignment. So they pay the difference between their bid and this surplus. For instance, consider an auction for two items, where bidders submit the following bids

- Bidder 1: $30 for 1 Item, $55 for both
- Bidder 2: $20 for 1 item
- Bidder 3: $15 for 1 item, $20 for 2

In this example, bidder 1 wins both items. If the auctioneer didn't allocate the items to bidder 1, the most he could make is $20 for bidder 2 and $15 from bidder 3. Bidder 1 will pay $35 for the two items. With VCG prices, sincere bidding is a weakly dominant strategy in a sealed bid auction.

iii. An Example of VCG prices in this auction

Consider the following CCA auction with only one category:

Round 1:

Price of Good	Bidder 1 Bid	Bidder 2 Bid	Bidder 3 Bid	Total Supply	Excess Demand
5	3 units	3 units	3 units	3 units	6
6	3 units	2 units	1 unit	3 units	3
7	2 units	1 unit	0 units	3 units	0

In the supplementary bid round each bidder submits the following package bids

- Bidder 1: 20 for 3, 14 for 2, 7 for 1.
- Bidder 2: 17 for 3, 13 for 2, 7 for 1.
- Bidder 3: 15 for 3, 12 for 2.

The final allocation is bidder 1 gets 2, bidder 2 gets 1, since that provides an as-bid value of 21. If you allocated both those goods to bidder 2, then bidder 2 would have been willing to pay an addition 10. If you allocated 1 of those goods to bidder 2 and one to bidder 3, the opportunity cost would have been 12 (13-7 is bidder 2s contribution, bidder 3 has an active bid of 6 for 1 unit from the clock phase). If you allocated both of those goods to bidder 3 you would also have gotten 12. So bidder 1 pays 12 for these two items. A similar, (but easier) calculation shows that bidder 2 pays 6 for the one item he wins.

iv. Core Pricing

Instead of using VCG prices, this auction selects the bidder-optimal core point closest to the VCG prices. An allocation of payoffs $\pi = (\pi_1, \pi_2, \ldots \pi_n)$ where π_i is the payoff bidder i receives, is in the core if the following holds. Let 0 denote the seller. Define the coalitional value function:

$$W(S) = \begin{cases} \max_{x \in X} \sum_{i \in S} U_i(x_i), 0 \in S \\ 0, \quad otherwise \end{cases}$$

π is in the core if $\sum_{i=0}^{n} \pi_i = W(N)$, where N is the set of all bidders, and $w(S) \leq \sum_{i \in S} \pi_i$ $\forall S \subseteq N$. A core allocation leads to more reasonable looking prices when there are complementarities, since they eliminate all outcomes where a group of bidders is able to offer the auctioneer more than the final allocation. For instance, consider the following preferences.

	A	B	AB
Bidder 1	0	4	4
Bidder 2	4	0	4
Bidder 3	0	0	4

The VCG allocation is bidder 1 gets A, bidder 2 gets B, but both bidders pay 0. This seems odd, since bidder 3 is willing to pay more than 0 for AB. The final payoff vector is $(0,4,4,0)$. In the above notation $W(\{0,3\}) = 4 > 0$. The closest to Vickrey core allocation is bidder 1 and 2 both pay 2, receiving a payoff of 2 each. Then $\pi = (4,2,2,0)$ so $W(\{0,3\}) = 4 = \pi_0 + \pi_3$. The core of any auction is non-empty. In all cases considered in this paper, the VCG outcome lie in the core given bidder's true preferences, since bidders have substitute preferences (Cramton 2009).

By using these prices, instead of uniform prices, it's expected that there will be less demand reduction. Since bidders are paying prices that are similar to VCG prices, they have incentives to bid truthfully (Day, Milgrom 2008), since their bids no longer directly effects the price they pay (Cramton 2009).

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vi. Activity Rules

In a clock auction it is important to have an activity rule, a rule that constrains bids in order to encourage bidders to submit bids in every round. This ensures bidders reveal information about their values in every round, aiding in price discovery. Without a well constructed activity rule, bidders could attempt to “bid snipe”, by only submitting bids near the end of the auction in order to win items at lower prices.

The simultaneous multiple round auction uses an activity rule based on eligibility. Each item is assigned some amount of eligibility points. Bidders can only submit for items that have at most the eligibility of the items bid on in the previous rounds. For instance if there is an auction for two units of item A, which is worth 30 eligibility points, and B, which is worth 20 eligibility points, a bidder who bids on only item B is not eligible to bid for A or AB in subsequent rounds.

Instead of an eligibility point rule, Ausubel, Cramton and Milgrom (2006) propose an alternate rule. Instead of constraining bids by eligibility points, bids are constrained based on the preferences revealed in previous rounds. Bids in round t are constrained by:

$$(p_t - p_s) \cdot (x^t - x^s) \leq 0 \quad for\ all\ s < t$$

Where p_t is the price at time t and x^t is the quantity at time t. In the supplementary bid round, bids for a package q are constrained by

$$b(x) \leq B(x^t) + p_t \cdot (x - x^t) \quad For\ all\ t$$

Where b(x) is the bid submitted for package x, and $B(x_t)$ is the maximum bid for package q_t (including bids in the clock phase). After rearranging the terms in this, the motivation is clear.

$$b(x) - p_t \cdot x \leq B(x^t) - p_t \cdot x^t$$

At time t, when the bidder decided to bid on package x^t instead of package x, they revealed a that at prices p_t, the profit from winning package x^t must be greater than the profit from winning package q.

This activity rule constrains bidders by preference instead of by the size the packages they bid on. With eligibility points, bidders can “park” on items they do not intend to win, instead bidding on items only to maintain eligibility in later rounds. For instance, if there is a item bidders expect to be highly demanded at the current price, bidders can bid on this to maintain eligibility without risking winning the item. This leads to less price discovery, and discourages straightforward bidding (Cramton 2009).

While it seems reasonable to assume that this auction will perform well, and it has performed well in both experimental tests and early field tests, the equilibrium analysis has not previously been performed. Other auction designs, for instance the Ausubel ascending auction (Ausubel 2004), are dynamic auctions which posses efficient equilibria (in the case considered in this paper, sincere bidding is the unique strategy that survives iterated elimination of weakly dominated strategies in the Ausubel auction).

This analysis will demonstrate that there are equilibrium strategies in this these simple cases that achieve an efficient allocation. This paper considers the case with n bidders and M homogeneous goods, and demonstrates that sincere bidding, bidding up to your value, is a equilibrium strategy.

3. The Homogeneous Goods Case

i. The Model

Consider an auction for M homogeneous goods with n players. Let X_i denote the consumption set for player $i \in \{1,2, \dots n\}$. A player bids on an quantity x_i, and the final allocation must be a set of x_i that satisfies the feasibility constraint $\sum_{i=1}^{n} x_i \leq M$. Consider goods to be indivisible (i.e. $X_i = \{0,1,2, \dots M\}$). Assume bidders have quasilinear utility (a strong, but standard assumption of the theory), where $U_i(x_i) - y_i$ is the utility for a package x_i at price y_i. Define $u_i: X_i \to \mathbb{Z}$ as the marginal value of quantity x, so $U_i(x_i) = \sum_{q=1}^{x_i} u_i(q)$. At each round, the auctioneer announces price $p_t = t$. By treating both marginal utilities and prices as integer values, this auction is a discrete game, and by limiting bids to integer values avoids the case where the price is below a bidder's value in one round and above a bidder's value in the next. Bidders then submit a bid for a quantity x_i at these prices. This procedure continues until $\sum x_i \leq M$(supply exceeds demand). Bidders then are allowed to submit another set of package bids, denoted by $b(x_i)$, and these bids and the bids from the clock phase are used to determine winners and final prices.

Bids in the clock phase are constrained by a monotone activity rule, specifically a bid at time t is constrained by

$$x_i^t \leq x_i^{t-1}$$

$$For\ all\ i = 1,2, \dots\ n$$

$$For\ all\ t$$

Bids in the supplementary bid phase are constrained by:

$$b(x_i) \leq B(x_i^t) + p_t(x_i - x_i^t)$$

Where t is the period where the bidder was last eligible to bid on x_i. All packages smaller then the final clock package are constrained by $B\left(x_i^f\right) + p_f\left(x_i - x_i^f\right)$ where f is the final clock round.

Lemma 1: *When all goods are in the same category, both an eligibility point activity rule and a revealed preference rule are the same as a monotone activity rule.*

The final allocation is the allocation $A_i = \{x_1, x_2, \dots x_n\}$ which solves $\max_{A_i} \sum B_i(x_i)$, where $B_i(x_i)$ denotes the highest bid submitted by player i for the package x_i.

ii. Homogeneous Goods With Private, Non-increasing Marginal Value

In this case, sincere bidding is an equilibrium. Specifically, the strategy where each bidder i at time t submits a bid for quantity $Q_i(t) = \min\{\, argmax_{x \in [0, Q_i(t-1)]}\{U_i(x) - tx\}\}$, meaning bidders bid for the item until the clock price for it exceeds their value for it. In the supplementary bid round, if there is no excess supply, bidders bid their value on the quantity they bid on in the final round of the clock phase, and on all other packages. If there is excess supply in the supplementary bid, then they should bid the bid suggested in Proposition 2 of Cramton (2009).

"***Proposition 2.*** *If the clock stage ends with excess supply, then a winner can guarantee that it wins its clock assignment by raising its bid on its clock package by the value of the unsold lots at the final clock price*" (Cramton 2009).

Specifically, denote the value of unsold lots at the final clock price by Ω_i. For each unit of excess supply, submit a bid for a package of $Q_i(t) + n$= Ω_i +$\sum_{j=Q_i(T)+1}^{Q_i(T)+n} u_i(j)$, for all $0 \le n \le M - Q_i(T)$. For example, if a bidder's marginal utility for one good is 10 and for the second good is 5, that player will bid on two goods until round 6, and will bid on one good until round 11. In the supplementary bid phase, bidders will bid 10 for 1 good and 15 for both.

Lemma 2: *If all bidders bid sincerely, then you can never pay more than your utility for the final clock package.*

If all players play this strategy, the final allocation will allocate the goods to the players with the M highest utilities, and they will pay the VCG prices.

Theorem 1: *Sincere bidding is a Ex-Post Perfect Nash Equilibrium*

Proof in Appendix

Unfortunately, this is not a weakly dominant strategy. For instance, consider the two bidder case, 2 goods case. If a bidder decided to play the strategy bid for both goods until excess demand dropped to 1, and then not submit any supplementary bids, the other bidder should reduce their quantity immediately.

So, in the homogeneous goods case, this auction design has an equilibrium which leads to an efficient allocation. Further, the combinatorial clock auction removes some of the equilibria in cases where the simultaneous multiple round auction would encourage demand reduction. For instance, in a uniform price auction with two bidders and two goods, bidders can each only bid on one good, reducing their demand for the second good to 0. This allows them to each win a good at price 0. This is not an equilibrium in the combinatorial clock auction.

Lemma 3: *There are 2 homogeneous goods and 2 bidders.* $u_i(1) \neq u_j(1)$. *Then, the "low price equilibrium" where both bidders reduce their demand to 1 unit in the first period, is not an equilibrium in the combinatorial clock auction.*

There results suggest that the postulated advantages of this design over the simultaneous multiple round auction are accurate. Bidders have strong incentives to reveal their values in the homogeneous goods case.

4. The Heterogeneous goods Case

i. Heterogeneous Goods

While the homogeneous goods case considered in this paper provides a good base case, what is far more interesting is the heterogeneous goods case.

Instead of one category of homogeneous goods, consider N categories, each category containing some quantity of homogeneous goods, and maintain all other aspects of the original model (specifically, the monotone activity rule). Both experimental and actual spectrum auctions are structured like this, and the design has performed reasonably well.

Clearly the general version of this case would be very difficult, and quite possibly impossible to analyze. So, instead consider the case where all goods satisfy the gross substitutes property, when the price of some goods are increase and the prices of all other goods are held constant, then demand for the goods whose prices are held constant should weakly increase. It seems reasonable that in this case, the combinatorial clock auction should have an equilibrium where bidders bid sincerely, or at least an equilibrium which yields an efficient outcome. Other auction designs, for instance, the extension of the Ausubel auction for heterogeneous goods (Ausubel 2006), possess similar equilibrium in this case. If this auction does not possess equilibrium where bidders bid somewhat sincerely in this simple, it seems reasonable that bidders will not bid sincerely in practice. This would limit price discovery in the clock phase and suggest that this auction does not have some of the postulated advantages.

ii. Activity Rules

If this auction has an equilibrium where bidders bid sincerely, it may not be preserved under some activity rules. Specifically, an activity rule based on eligibility points may make it optimal for bidders to deviate from sincere bidding in order to keep the price for items they want to win low or simply to maintain high levels of eligibility, so they can bid more flexibly later on in the auction (Cramton 2009).

iii. Point-Based Activity

The way to extend the monotone activity rule for homogeneous goods that seems most natural is through an eligibility point rule. As detailed above, this rule gives each item a "size" and bidders cannot bid on packages that were larger than the package bid on in the previous round. Specifically, if n different categories of goods are being auctioned, the auctioneer also announces a vector E, $E \in \mathbb{R}_+^n$ such that bidders can only bid on packages X where $(X \cdot E) \leq (X_{t-1} \cdot E)$. This rule has been used in almost all spectrum auctions. In the supplementary bid round, let bidders only raise bids on a package up to the price the package was at the last round they were eligible to bid on a package of that size (this is basically the version of the rule used in an auction for L Band in the UK (Ofcom 2007)). Their bids are unconstrained on any package they were eligible to bid on after the final clock round.

While this rule may seem like the natural way to constrain bidding, it is easy to construct an example where by bidding their true demand in early rounds, a bidder constrains himself out of bidding sincerely in later rounds. For instance, consider an auction for goods A and B. Consider a bidder with the following preferences:

A	B	AB
4	3	4

Assume the bidder bids on the most profitable package they can still bid on in each round. Now consider a clock auction where item A has size 1, B has size 2, that proceeds as follows for this bidder.

Price (A,B)	Bidder's Unconstrained Demand	Bid
(1,1)	A	A
(2,1)	A,B	A
(3,1)	B	A

So, by bidding their true demand in the early rounds, this bidder has constrained themselves out of bidding for their most profitable package in later rounds, not potentially lowering their payoff, but also lowering the efficiency of the final allocation. On the other hand, if the bidder instead bids on the "largest" profitable package, they can bid their demand every round and bid their true demand in the supplementary bid round, leading to the VCG outcome. An eligibility point rule provides the bidder with strong incentives so submit a set of bids with provide very little information on bidder's true valuations, or which packages are demanded. This hurts the ability of bidders to engage in price discovery during the clock phase, removing many of the benefits from using a dynamic auction in this situation.

One may think that this problem is due to "miscalibration" of the points. The auctioneer could believe that there was some way to redo the eligibility point totals so that bidders who bid truthfully in early rounds would not constrain themselves from bidding truthfully in later rounds. But, as the following result demonstrates, even with simple preferences, there could be no eligibility point system that would allow bidders to submit truthful bids at every round of the auction, and that bidding sincerely in this case leads to an inefficient outcome.

Definition: A bidder's preferences are said to satisfy the *Gross Substitutes Property* if for any two price vectors p and p' such that $p^{'} \geq p$ and demand $q_i(\cdot)$ is single valued at p and p', then $q_i^k(p^{'}) \geq q_i^k(p)$ for any commodity such that $p'^k = p^k$.

Theorem 2: *Assuming bidders bid on the largest package (in terms of eligibility points) that solves* $argmax\ U_i(X) - p_t \cdot X$ *such that X is feasible in each round, and* $max\ (U_i(X), Constraint\ on\ X)$ *in the supplementary bid round, there is a set of goods and bidders with substitute preferences such that no set of eligibility points lead to an efficient outcome.*

(Proof is in appendix 2)

Consider an auction for two goods A and B and two units of each. Consider the following quasilinear preferences.

	A	AA	B	BB	AB	AAB	ABB	AABB
Bidder 1	6	6	3	6	6	6	6	6
Bidder 2	7	14	10	10	10	14	10	14
Bidder 3	2	4	1	2	3	5	4	6

The VCG outcome is that bidder 1 receives BB, and bidder 2 receives AA. Bidder 1 pays 2, bidder 2 pays 4 (this is also a Walrasian equilibrium). But, bidder 1 wants to substitute one unit of A into two units of B, while bidder 2 (roughly) wants to substitute 1 unit of B for 2 units of A. If bidders bid sincerely in early rounds, they are constrained from bidding sincerely in later rounds. This leads to an inefficient outcome. For instance, let E_a be the eligibility point for A, E_b be the eligibility points for B

$$E_b > E_a, 2E_a > E_b$$

Price (A,B)	Bidder 1 Bid (Demand)	Bidder 2 Bid (Demand)	Bidder 3 Bid (Demand)
(1,1)	A	AA	AABB
(2,1)	A	AA	AABB
(3,1)	A	B	BB
(3,2)	A	B	$\emptyset$

Supplementary bids:
Bidder 1- A: 6

Bidder 2-A: 7, AA: 6, B: 10
Bidder 3: Bids demand
Final Allocation: A to 1, B to 2, AB to 3, bidder 1 pays 2, bidder 2 pays 1, bidder 3 pays 0.
So bidders A receives a payoff of 4, bidder 2 receives a payoff of 9, bidder 3 receives a payoff of 3.

This is not efficient. Consider an alternative allocation where bidder 1 receives BB, bidder 2 receives AA, and bidder 3 receives 3 from bidder 2. This gives bidder 1 a payoff of 4, bidder 2 receives a payoff of 10, and bidder 3 receives a payoff of 3, which is an outcome which dominates the outcome of this auction.

This auction proceeds in a similar manner for any set of eligibility points. There is always a point in the auction where bidders can no longer bid their true demand (it is in the last clock round of the above auction, bidder 1 would like to bid on B, but can only bid on A), and this constraint makes it impossible to place the bid they would need to place to win their desired package.

iv. Revealed Preference Activity Rule

Another way of extending the monotone activity rule to the heterogeneous goods case is to constrain bidding based on revealed preference. This rule, first developed in Ausubel, Cramton, Milgrom (2006), forces bidders to bid in a way that is consistent with the laws of revealed preference (basically requires bidders to bid along some downward sloping demand curve). As stated earlier, bids must satisfy the following inequalities. In the clock phase, a bid must satisfy:

$$(p_t - p_s) \cdot (q_t - q_s) \leq 0 \quad for\ all\ s < t$$

Where p_t is the price at time t and q_t is the quantity at time t. In the supplementary bid round, bids for a package q are constrained by

$$b(q) \leq B(q_t) + p_t \cdot (q - q_t) \quad For\ all\ t$$

Where $B(\cdot)$ denotes the largest bid made for package q_t, and b(q) is the supplementary bid for package q.

For any utility maximizing consumer, this property is satisfied, so as long as a bidder bids their true demand in every round they have no problem bidding their demand in future rounds. So the rules of the auction no longer punish bidders for bidding their true demand in each round. For instance, consider the previous set of preferences.

	A	AA	B	BB	AB	AAB	ABB	AABB
Bidder 1	6	6	3	6	6	6	6	6
Bidder 2	7	14	10	10	10	14	10	14
Bidder 3	2	4	1	2	3	5	4	6

Now consider bidders bidding their demand in every round. There is no longer a clear way to choose the package demanded when demand is not single valued, so this is not a unique price path (I have selected the path that yields Walrasian prices in the final round. While not all price paths do this, bidders will always be able to state their true preferences in the supplementary bid round and the auction will select the VCG outcome).

$$E_b > E_a, 2E_a > E_b$$

Price (A,B)	Bidder 1 Bid (Demand)	Bidder 2 Bid (Demand)	Bidder 3 Bid (Demand)
(1,1)	A	AA	AA
(2,1)	BB	AA	∅

Supplementary bids: All bidders bid demand. The final allocation is the VCG allocation.

It seems reasonable to ask if there is an equilibrium for this auction where bidders bid their true demand in every round of the clock stage.

v. The Model

Assumption 1: Bidders have quasilinear, integer valued utility where $U_i(x_i) - y_i$ is the utility for a package x_i at price y_i, and $U_i: X_i \to \mathbb{Z}$.

Assumption 2: Preferences satisfy that for any prices p and p' and demand q_i, $(q_i(p) - q_i(p')) \cdot (p - p') \leq 0$.

Assumption 3: Bidders preferences satisfy the substitutes property.

Assumption 4: Full support: any set of substitute preferences could occur.

Model this as a two stage auction. In the first stage bidders report a set of valuations $\Pi: X \to \mathbb{R}_+$ satisfying the substitutes condition to a "proxy bidder." The auctioneer then has the proxy bidder report values along some non-decreasing price path p(t). In the second stage bidders report another set of values $\Pi^*: X \to \mathbb{R}_+$, also satisfying the substitutes condition, which are consistent with respect to the revealed preference constraints from the first stage (so $\Pi^*(x) \leq \Pi^*(x^t) + p_t \cdot (x - x^t) \ \forall t$). Bidders are doing something similar to this in the above model (due to the revealed preference constraint, they must bid in a way that is consistent with some set of preferences), but they are not limited to reporting substitute preferences.

vi. The Result

Theorem 3: *In the combinatorial clock auction with a revealed preference activity rule, there is an ex-post perfect equilibrium where the bidder reports their true valuations in both the clock phase and the supplementary bid phase*

(Proof in appendix)

In this case, under the revealed preference activity rule, bidders bid their true demand in every round of the clock phase. This allows them to fully reveal their preferences in the supplementary bid round, which in turn guarantees and efficient outcome.

This result suggests that a revealed preference activity rule is encourages bidders to bid truthfully, allowing the bidders to observe relevant information about the other bidders valuations. As these auctions become large (an auction for n different goods has 2^n possible packages) it becomes more difficult for a bidder to decide which packages to bid on. By providing incentives for other bidders to reveal information about their preferences, this problem becomes more manageable for all the bidders.

5. Conclusion

This paper has demonstrated that in a simple case, the combinatorial clock auction is able to induce bidders to bid sincerely, producing an efficient outcome. Since the homogeneous goods case is an important case of this auction, and there are other designs where the homogeneous goods case can achieve efficient outcomes, it would be troubling if this design did not have an efficient equilibrium.

This result speaks to the advantages of this design over the simultaneous multiple round auction. Even in this simple case, this design theoretically performs better then the simultaneous multiple round auction in some circumstances. As the environment becomes more complex, it seems reasonable that the advantages of this design will become even more pronounced.

This analysis also reveals that a point-based activity rule should probably not be used to constrain bidding. Since an eligibility point rule can provide incentives for bidders with even simple preferences to misrepresent their preferences, possibly forcing bidders to obscure their preferences by bidding on the largest profitable package, which provides the other bidders with almost no information. This hinders price discovery, removing many of the benefits of a dynamic auction.

A revealed preference rule does not have these problems. The rules of the auction no longer penalize bidders who bid sincerely. Those bidders are able to bid their true demand in each round, providing far more information about how they value the different packages.

The revealed preference activity rule is by no means perfect. Bidders have found it more difficult to understand and overly constraining (Cramton 2009). Perhaps a hybrid rule like the Eligibility Point Safe Harbor rule proposed in Ausubel, Cramton (2011), is a solution. This rule allows bidders to submit bids satisfying revealed preference. But it also allows them some additional flexibility in the clock round, since they can deviate to bid on packages that violate revealed preference that they still have the eligibility for.

Whatever rule is used, it is very important that it does not penalize bidders for bidding truthfully, like the eligibility point rule does. A rule like revealed preference, that allows bidders to bid their demand in each round, is very important to encourage price discovery in a dynamic auction like this.

Appendices

Appendix 1: Proofs of Lemma's and Theorems

Lemma 1: *When all goods are in the same category, both an eligibility point activity rule and a revealed preference rule are the same as a monotone activity rule in the clock phase.*

Proof: For eligibility points, this follows directly. Since all goods are in the same category, they all have the same number of eligibility points, so this is exactly the same as a monotone activity rule.

For a revealed preference rule, bids in the clock phase are constrained by $(p_s - p_t) \cdot (x_s - x_t) \leq 0$. Since prices are monotonically increasing $p_s - p_t$ is negative for all s<t. So $x_s \geq x_t$. Specifically, $x_{t-1} \geq x_t$, which is the monotone activity rule.

Lemma 2: (Justification of Ω_i). If all bidders bid sincerely, then you can never pay more than your utility for the final clock package.

> There was no excess demand in the final clock round, so for each bidder the marginal utility of winning any additional items must be less then T (the final clock round) for each item. But since each bidder bids sincerely, by winning the clock package they only displace bidders who dropped from those items earlier than the final clock round. So these bidders have submitted bids less then T, so the opportunity cost of assigning additional units to them must be less then T, which is less than or equal to your marginal utility for each of these items, so you will never pay more than your value for the final clock assignment. Specifically bidding Ω_i, can never cause you to pay more than your value for the final clock assignment. In fact, it can never cause you to pay more than you would have paid if you bid $U_i(Q_i(T))$ and won the package.

Theorem 1: Sincere bidding is a Ex-Post Perfect Nash Equilibrium

Proof:

We can view VCG prices as follows. Consider a bidder i winning a quantity Q. If he instead won Q+1, then he causes bidder j to win 1 less unit (unless nobody else is bidding on that unit, then he pays 0 for the additional unit). Let $P(\cdot)$ be the Vickrey price. $P(Q+1) - P(Q) = B_j(Q_j + 1) - B(Q_j)$, since if we remove bidder i, then bidder j wins the additional unit. We can view this as bidder j's as bid marginal value for that extra item.

Consider any round s and any history of bids. Consider the following deviations from the prescribed strategy. Since this is an incomplete information game, no other bidders can respond to your deviation.

Deviates to win the same number of items

Suppose bidder i deviates. He wins the same number of units (n) that he would win. Each other bidder is always bidding sincerely in all future periods, so they bid up to the same value for each quantity. So in order to win n units, he must displace the same bidders. So he would pay the same price for these n units regardless of any deviation.

Deviates to win more units

In order to win more items, bidder i must deviate from sincere bidding at some point by bidding for more items that prescribe at some point in the clock phase. If he does not, then the only way to win a larger set of item than the final assignment without deviating in the supplementary bid round, is to submit a bid for the additional items that is higher than your marginal utility for them, since when he bid sincerely, he bids his marginal value, or the constraint for all items above the final clock price, and if he is already bidding the constraint he cannot deviate to bid more for those items. But since he didn't win these items when bidding sincerely, the bidder who won each of them was willing to pay more than his marginal utility for them, so he wins it for either negative or zero profit.

So for this to be a profitable deviation, at least one clock round t, where $Q_i(t) < Q_i(t-1)$ the bidder bids for Q, $Q_i(t) \leq Q \leq Q_i(t-1)$ instead. By making this deviation, he must anticipate winning some of these additional items because bidding on additional items cannot affect price of the smaller packages. So assume that submitting this you can win some number n of additional units $n \leq Q - Q_i(t)$, by making this deviation, you must anticipate winning some of these additional units.

$\big(U_i(Q_i(t)+n)\big) - t(Q_i(t)+n)$

$=U_i\big(Q_i(t)\big) - tQ_i(t) + \sum_{j=1}^{n} u_i(Q_i(t)+j) - tn$

$\leq U_i(Q_i(t)) - \ tQ_i(t)$, since otherwise he would have bid on these additional units while bidding sincerely. Therefore:

$$\sum_{j=1}^{k} u_i(Q_i(t)+j) - tn \leq 0$$

But winning these items at price t or higher cannot be profitable. So, as long as there is excess demand, by winning any of these additional items, he displaces bidders who have are have marginal utility of at least t for them, since all other bidders are bidding sincerely in round t. When these bidders reduce their quantity from $Q_j(t)$ to some other quantity $Q_j{}'$, in the supplementary bid round they will place a bid for $Q_j(t)$ of at least $B_j\left(Q_j{}'\right) + t(Q_j(t) - Q_j{}' - i)$ for each i s.t. $Q_j(t) - i \geq Q$. So the cost of winning these items is at least t for each, so this cannot be a profitable deviation.

If there is excess supply after you submit this bid, than he would win those additional items anyway as long as no other player has a higher marginal utility for winning them. He have submitted a bid of Ω for $Q_i(t)$, and $\Omega_i + \sum_{j=Q_i(T)+1}^{Q_i(T)+n} u_i(j)$ for all additional units. Since all bidders are bidding sincerely, they have submitted bids of $\Omega_i + \sum_{j=Q_k(T)+1}^{Q_k(T)+n} u_k(j)$ for additional units. Each bidder must receive at least his clock allocation, since there is no smaller allocation that can exceed, since all bids on smaller packages are constrained by $\Omega_i - some\ constant$. So if bidder i losses any larger package, then the price for each unit of the larger package must be greater than his marginal utility for that additional unit, so deviating to win these additional units cannot be profitable.

Deviates to win fewer units

Suppose the bidder deviates and win k fewer units then he would win without deviation (n). But since all other bidders are bidding sincerely in all rounds after t, the same n-k bid up to the same values for the each quantity. So the same bidders are displaced, so they pay the same price for those n-k items. So the price he would pay for those additional k units must not be profitable (moreover the price he would pay for each of those units must not be profitable, because otherwise he would want to displace a bidder and win an additional one). For some of these items to not be profitable, then these n items must be won due to an insincere bid, since with sincere bidding you never pay more than your utility. Consider the following possibilities:

Suppose there was excess demand in the s-1st round. Then there must have been excess demand in all previous rounds. Consider last round where bidder i did not bid sincerely (round s or before). There is an allocation, using only bids from round s, where the total as-bid value for the other bidders is ((M-n)+j)s, for some $n \geq j \geq 1$. There must be no excess demand in the following round, since if there was, and all bidders bid sincerely for the rest of the auction, then bidder i could not win an unprofitable package (since there must be a group of bidders willing to offer at least M(s+1) for all the items). So if there is no excess demand, bidder i has submitted a bid for $Q_i(s+1)$ units at price s+1. In the supplementary bid phase, all other bidders will submit bids for packages smaller then, or the same size as $Q_j(s)$ or smaller at price at least s for each unit, so there must be excess supply. Then the bidder would submit a bid of the maximum allowable bid for this final clock package, and bids of his marginal utility for each additional package. If he wins this final clock package, he pays the same price he would always pay for it, which must be profitable, since the bids for items in it are constrained by s, and he values every item in at a price greater than s. So he will never want to win a smaller package, since the opportunity cost of winning each of the units of the smaller package is the same as the cost of winning them when winning the larger package, and every unit of the larger package he wins is profitable.

What if there was no excess demand in round s-1?

If there is no excess supply, by proposition 1 of Cramton[1], the supplementary bid round cannot alter the final assignment, and he has no control over other bidders bids, so he there are no deviations to consider.

If there is excess supply, then he wins the clock package if he bids sincerely in the supplementary bid round (by proposition 2 from Cramton). If the final clock package (Q) was not $Q_i(T)$ (the package he would have bid on in that round if he bid sincerely), a smaller package may be more profitable. But he is constrained by $B_i(Q) - nT$ on any package n units smaller, and no other bidder is willing to offset that, since otherwise they would have bid for those additional items in the final clock round. If a larger package was more profitable, then he would have won it, since he is bidding his marginal utility for each additional unit in addition to the final clock package.

Lemma 3: There are 2 homogeneous goods and 2 bidders. $u_i(1) \neq u_j(1)$. Then, the "low price equilibrium" where both bidders reduce their demand to 1 unit in the first period, is not an equilibrium in the combinatorial clock auction.

There are two homogeneous goods, and two bidders with non-increasing marginal values. If both player play the strategy where they reduce their demand to one good, win it and pay zero, the players have a profitable deviation (assuming bidders cannot bid on larger packages after bidding on smaller ones). Instead of reducing demand, a bidder can simply bid for both goods. They will still win one good for zero, but now the bidder with the higher marginal utility for one good will win the second good, and pay the other bidder's marginal utility, giving him a positive profit

[1] "**Proposition 1:** If the clock phase ends with no excess supply, than the supplementary bid round cannot alter the final clock assignment" (Cramton 2009)

Theorem 3: *In the combinatorial clock auction with a revealed preference activity rule, there is an ex-post perfect equilibrium where the bidder reports their true valuations*($\Pi = U$) *in both the clock phase and the supplementary bid phase*

Proof:

Stage 2:

Given that you bid $\Pi = U$ in stage 1, it is weakly dominant to bid $\Pi^* = U$ in stage 2. Since this was weakly dominant in the unconstrained VCG auction, it is also weakly dominant in this constrained game.

Stage 1:

Assume all other bidders are bidding $\Pi_\text{i} = U_i$. Then they will also bid $\Pi_i^* = U_i$ in the next round. If you deviate from bidding U in this round, you are constrained from bidding U in the supplementary round, so if all other bidders are bidding their true valuations in the supplementary bid round, then whatever your best response is in the 2nd round, it cannot provide a higher payoff then not deviating and bidding U did. Since no bidder can respond to you not bidding $\Pi = U$ in stage, there is no way to make every other bidder not bid their true valuation in the supplementary bid round. So this is an ex-post perfect Nash equilibrium.

Appendix 2: Proof of Theorem 2

Theorem 2: Assuming bidders bid on the largest package that solves max $U_i(X) - p_t \cdot X$ such that X is feasible in each round, and max $(U_i(X), Constraint\ on\ X)$ in the supplementary bid round, there is a set of goods and bidders with substitute preferences such that no set of eligibility points lead to an efficient outcome.

Proof:

First note, since bidders have quasilinear utility, the only efficient outcomes are outcomes where the assignment is the value maximizing assignment. If the value maximizing assignment is not won, bidders (and possibly the seller) can trade to receive it, and those trades will add value to the total value of the group of bidders and sellers. Side-payments can then be used to make all bidders at least as well off as they were before trades, and this allocation will Pareto dominate the original allocation.

Consider an auction for 2 categories of goods, each with 2 units and bidders with the following preferences:

Bidder	A	AA	B	BB	ABB	AAB	AB	AABB
1	6	6	3	6	0	0	0	0
2	7	14	10	10	0	0	0	0
3	2	4	1	2	4	5	3	6

The VCG outcome is bidder 1 gets BB, bidder 2 gets AA. Bidder 1 pays 2, bidder 2 pays 4.

In any efficient outcome, bidder 1 will receive BB, bidder 2 will receive AA.
Consider the following Cases
Let E_a be the eligibility point for A, E_b be the eligibility points for B

Case 1:

$$E_b > E_a, 2E_a > E_b$$

Price (A,B)	Bidder 1 Bid (Demand)	Bidder 2 Bid (Demand)	Bidder 3 Bid (Demand)
(1,1)	A	AA	AABB
(2,1)	A	AA	AABB
(3,1)	A	B	BB
(3,2)	A	B	∅

Supplementary bids:
Bidder 1- A: 6
Bidder 2-A: 7, AA: 4, B: 10
Bidder 3: Bids demand
Final Allocation: A to 1, B to 2, AB to 3, bidder 1 pays 2, bidder 2 pays 1, bidder 3 pays 0.
So bidders A receives a payoff of 4, bidder 2 receives a payoff of 9, bidder 3 receives a payoff of 3.

This is not efficient.

Case 2:

$$E_b > E_a, 2E_a = E_b$$

The clock phase is the same as above. In the supplementary bid phase, supplementary bids are:
Bidder 1- A: 6
Bidder 2-A: 7, AA: 14, B: 10
Bidder 3: Bids demand
Final Allocation: A to 1, B to 2, AB to 3, bidder 1 pays 3, bidder 2 pays 1, bidder 3 pays 0.

This is not efficient.

Case 3:

$$E_b > E_a, 2E_a < E_b$$

Price (A,B)	Bidder 1 Bid (Demand)	Bidder 2 Bid (Demand)	Bidder 3 Bid (Demand)
(1,1)	A	AA	AABB
(2,1)	A	AA	AABB
(3,1)	A	AA	BB
(4,1)	A	AA	BB
(5,1)	A	AA	BB
(6,1)	A	AA	BB
(7,1)	∅	AA	BB

Supplementary Bids:
Bidder 1: 6 for A
Bidder 2: 7 for A, 14 for AA
Bidder 3: Bids demand
Final Allocation: Bidder 2 gets AA pays 8, bidder 3 gets BB pays 0.

This is not efficient.

Case 4:

$$E_b < E_a, E_a < 2E_b$$

Price (A,B)	Bidder 1 Bid	Bidder 2 Bid	Bidder 3 Bid
(1,1)	A	AA	AABB
(2,1)	A	AA	AABB
(3,1)	A	B	BB
(3,2)	A	B	∅

Supplementary bids:
Bidder 1: A-6, B-3
Bidder 2: B-10, AA-4, A-2, BB-2
Bidder 3: Bids demand
Final Allocation: 1-A, 2-B, 3-AB

Bidder 1 pays 2, bidder 2 pays 1, bidder 3 pays 0.

This is not efficient.

Case 5:

$$E_b < E_a, 2E_b = E_a$$

Price (A,B)	Bidder 1 Bid	Bidder 2 Bid	Bidder 3 Bid
(1,1)	A	AA	AABB
(2,1)	BB	AA	AABB
(3,2)	A	AA	∅
(4,2)	A	B	

Supplementary Bids:
Bidder 1: A-6, B-3, BB-6
Bidder 2: B-10, AA-8, A-4, BB-4
Bidder 3: Bids demand
Final Allocation bidder 1-A, bidder 2-B, bidder 3-AB
Bidder 1 pays 2, bidder 2 pays 1, bidder 3 pays 0.

This is not efficient.

$$E_b < E_a, 2E_b < E_a$$

Price (A,B)	Bidder 1 Bid	Bidder 2 Bid	Bidder 3 Bid
(1,1)	A	AA	AABB
(2,1)	BB	AA	AABB
(3,2)	BB	B	∅
(3,3)	BB	B	∅
(3,4)	∅	B	∅

Supplementary bids:
Bidder 1: A-2, B-3, BB-6
Bidder 2: A-3, AA-6, B-10, BB-4
Bidder 3: Bids demand
Final Allocation B to bidder 2, B to bidder 1, AA to bidder 3.
Bidder 1 pays 1, bidder 2 pays 3, and bidder 3 pays 0.
Payoffs: Bidder 1: 2, Bidder 2: 7, bidder 3: 4

This is not efficient.

So there is no combination of eligibility points that leads to an efficient outcome if bidders bid sincerely.

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